Asymptotic Minimax Risk for the White Noise Model on the Sphere

Proposed running head: Minimax Risk with Spherical Data

Jussi Klemelä
Rolf Nevanlinna Institute
University of Helsinki

Abstract

Estimation of an unknown function on the unit sphere of the Euclidean space is considered. The function is observed in Gaussian continuous time white noise. Uniform norm is chosen as a loss function and exact asymptotic minimax risk is derived extending the result of Korostelev (1993). The exact asymptotic minimax risk is given also for the L_2-loss applying the result of Pinsker (1982).

Key Words:

asymptotic minimax risk, exact constants in nonparametric smoothing, Gaussian white noise, spherical data, uniform norm.

1 Introduction

Let $S_d = \{x \in \mathbf{R}^{d+1} : \|x\| = 1\}$ be the unit sphere on the $d+1$ dimensional Euclidean space, where $d \geq 1$. Consider the observation of the form

$$dX(x) = f(x)dx + \epsilon dW(x), \; x \in S_d$$

where $\epsilon > 0$ is the noise size and $dW(x)$ is the Gaussian white noise in S_d, that is for measurable sets $B, B_1, B_2 \subset S_d$,

(i) $\mathcal{L}\left(\int_B dW(x)\right) = N(0, \mu(B))$ where μ is the Lebesgue measure of S_d,

(ii) if $B_1 \cap B_2 = \emptyset$, then $\int_{B_1} dW(x)$ and $\int_{B_2} dW(x)$ are independent.

The Lebesgue measure of the sphere can be defined with the help of the Lebesgue measure of $\mathbf{R}^{d+1}$, see Rudin (1986, page 175). It is a Borel measure, invariant with respect to rotations, and $\mu(S_d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$.

Let $\Lambda(\beta, L)$ be the class of Hölder functions

$$\Lambda(\beta, L) = \left\{ f : S_d \to \mathbf{R} \;\middle|\; |f(x) - f(y)| \leq L\delta(x,y)^\beta, \; x, y \in S_d \right\}$$

where $0 < \beta < 1$ and δ is the Riemannian metric of S_d, that is, $\delta(x,y) = \arccos(x'y)$. We define the discrepancy of an arbitrary estimator $\hat{f}_\epsilon$ and the true signal function f by the supremum norm

$$\|\hat{f}_\epsilon - f\|_\infty = \sup_{x \in S_d} |\hat{f}_\epsilon(x) - f(x)|.$$

Let $w(u)$, $u \geq 0$, be a loss function, i. e. a continuous bounded monotone function of u, $w(0) = 0$. Introduce the minimax risk

$$r_\epsilon = r_\epsilon(w, \beta, L) = \inf_{\hat{f}_\epsilon} \sup_{f \in \Lambda(\beta, L)} E_{\epsilon, f} w\left(\psi_\epsilon^{-1} \left\|\hat{f}_\epsilon - f\right\|_\infty\right)$$

where $\psi_\epsilon = (\epsilon^2 \log \epsilon^{-2})^{\beta/(2\beta+d)}$. The exact value of the asymptotic minimax risk is given in the next theorem.

Theorem 1 *Let* $A_{\beta,d} = \lim_{\kappa\to\infty} \|\kappa^{d/2} K_{\kappa,\eta}\|_2^{-2\beta/(2\beta+d)}$ *where* $K_{\kappa,\eta} : S_d \to \mathbf{R}$,

$$K_{\kappa,\eta}(x) = \left(1 - (\kappa\delta(x,\eta))^\beta\right)_+ \tag{1}$$

with $\kappa > 0$ *and* $\eta \in S_d$. *Let*

$$C_0 = L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{2d}{2\beta+d}\right)^{\beta/(2\beta+d)}.$$

Then

$$\lim_{\epsilon\to 0} r_\epsilon = w\left(C_0\right).$$

The constant $A_{\beta,d}$ can be calculated explicitily, because

$$\begin{aligned}
\|\kappa^{d/2} K_{\kappa,\eta}\|_2^2 &= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^\pi \left(1 - (\kappa\theta)^\beta\right)_+^2 \sin^{d-1}\theta\, d\theta \\
&= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} \left(1 - t^\beta\right)_+^2 \sin^{d-1}(\kappa^{-1}t)\kappa^{-1} dt \\
&\sim \mu_{d-1}(S_{d-1}) \int_0^1 \left(1 - t^\beta\right)^2 t^{d-1} dt \qquad (2)\\
&= \mu_{d-1}(S_{d-1}) \frac{2\beta^2}{(2\beta+d)(\beta+d)d}
\end{aligned}$$

where $\mu_{d-1}(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$, $d \geq 2$ and we use the convention $\mu_0(S_0) = 2$. Thus,

$$\lim_{\epsilon\to 0} r_\epsilon = w\left(L^{d/(2\beta+d)} \left(\frac{(\beta+d)d^2}{\mu_{d-1}(S_{d-1})\beta^2}\right)^{\beta/(2\beta+d)}\right).$$

Korostelev (1993) proved corresponding result for estimating regression function defined on the unit interval from n samples $X_i = f(i/n) + \epsilon\xi_i$, $\xi_i \overset{i.i.d}{\sim} N(0,1)$. Donoho (1994) presented results for the continuous time white noise model on the real line, and gave also evaluation of the asymptotic minimax risk for estimating derivatives of the signal function.

Let us also consider estimation with the L_2-loss assuming that the signal function belongs to the Sobolev ball. We will give exact asymptotic minimax constant applying the result of Pinsker (1982) to the spherical case.

Let $H_s(S_d)$, $s > 0$, be the Sobolev space of S_d. Then $f \in L_2(S_d, \mu)$ is in $H_s(S_d)$ if and only if

$$\|f\|_s^2 \stackrel{\mathbf{def}}{=} \sum_{k=1}^{\infty} \sigma_k^s \sum_{f_j \in E_k} < f, f_j >_{L_2(S_d,\mu)}^2 < \infty, \tag{3}$$

where $(\sigma_k)_{k=0}^{\infty}$ is the set of different eigenvalues of the Laplace-Beltrami operator Δ in increasing order, $(E_k)_{k=0}^{\infty}$ is the set of eigenspaces of Δ (E_k corresponds to σ_k for each k), and $(f_j)_{j=0}^{\infty}$ is an orthonormal basis of $L_2(S_d, \mu)$ consisting of real eigenfunctions of Δ (Giné, 1975). Let $\Sigma(s, M) = \{f \in L_2(S_d, \mu) : \|f\|_s^2 \leq M\}$, where $s > 0$ and $\|\cdot\|_s$ is defined in (3). Define

$$R_\epsilon(\Sigma(s, M)) = \inf_{\hat{f}} \sup_{f \in \Sigma(s,M)} E_{\epsilon,f} \left\| \hat{f} - f \right\|_{L_2(S_d,\mu)}^2 .$$

The exact value of the asymptotic minimax risk with the L_2-loss is given in the next theorem.

Theorem 2

$$R_\epsilon(\Sigma(s, M)) \sim \epsilon^{4s/(2s+d)} M^{d/(2s+d)} P_{s,d}, \quad \epsilon \longrightarrow 0,$$

where

$$P_{s,d} = \left(\frac{2s}{(s+d)(d-1)!} \right)^{2s/(2s+d)} \frac{(2s+d)^{d/(2s+d)}}{d}.$$

Previously for instance Hall, Watson, and Cabrera (1987) have studied kernel density estimation on the sphere and Hendriks (1990) has studied Fourier series density estimates on Riemannian manifolds. However, exact minimax constants have not been previously given in the case of spherical data.

In general, the analysis of kernel density estimators on the sphere involves more technical difficulties than the analysis of Fourier series estimators. This is due to the fact that Fourier methods allow more unified treatment of Euclidean and spherical cases. The kernel estimator is asymptotically minimax for the supremum norm loss and under Hölder smoothness whereas Fourier series estimator is asymptotically minimax for the L_2 norm loss and under Sobolev smoothness.

The proof of Theorem 1 is given in sections 2 and 3. The upper bound is proved in Section 2 using ideas presented in Korostelev (1993) and the lower bound is proved in Section 3 using ideas presented in Donoho (1994). The proof of Theorem 2 is given in Section 4.

2 The Upper Bound

We will give the proof for the upper bound for Theorem 1. Let us define the estimator

$$\hat{f}(x) = \sum_{i=1}^{N} \hat{a}_{Ni} I_{A_{Ni}}(x)$$

where

$$\hat{a}_{Ni} = c(\kappa) \int_{S_d} K_{\kappa,\eta(N,i)}(x) X(dx),$$

$K_{\kappa,\eta}$ was defined in (1), $c(\kappa)^{-1} = \int_{S_d} K_{\kappa,\eta} d\mu$,

$$\kappa = \left(\frac{C_0 \psi_\epsilon}{L} \right)^{-1/\beta},$$

$$N = \left[\chi^{-1} (\epsilon^2 \log \epsilon^{-2})^{-d/(2\beta+d)} \right] = \left[\chi^{-1} \psi_\epsilon^{-d/\beta} \right]$$

where $[b]$ is the greatest integer $\leq b$,

$$\chi = \frac{1}{D} \left(\frac{\gamma C_0 d}{L(2\beta + d)} \right)^{d/\beta}$$

where $\gamma > 0$ is arbitrary, D is a sufficiently large constant to be specified in (6), $\eta(N,i) \in S_d$, $i = 1, \ldots, N$, are asymptotically equidistant on S_d, and A_{Ni}, $i = 1, \ldots, N$, is a partition of S_d such that A_{Ni} is the set of those $x \in S_d$ for which $\eta_{N,i}$ is the closest member of the set $\{\eta_{N,1}, \ldots, \eta_{N,N}\}$ (ties can be broken arbitrarily). Thus

$$A_{Ni} \subset \{x \in S_d : i = \operatorname{argmin}_{i=1,\ldots,N} \delta(\eta(N,i), x)\}. \tag{4}$$

Let us start with the variation of the estimator.

Lemma 3

$$\lim_{\epsilon\to 0} P\left(\psi_\epsilon^{-1}\|\hat f - E\hat f\|_\infty > (1+\gamma)C_0\frac{2\beta}{2\beta+d}\right) = 0.$$

Proof. Let us denote $Z_\epsilon(x) = \hat f(x) - E\hat f(x)$. Then

$$\begin{aligned} D_\epsilon^2 &\stackrel{def}{=} Var\left(\psi_\epsilon^{-1} Z_\epsilon(\eta(N,1))\right) = \psi_\epsilon^{-2} Var(\hat a_{N1}) \\ &= \psi_\epsilon^{-2}\epsilon^2 c(\kappa)^2 \int_{S_d} K^2_{\kappa,\eta(N,1)}(x)d\mu(x) \\ &\sim \psi_\epsilon^{-2}\epsilon^2\kappa^d\mu_{d-1}(S_{d-1})^{-1}\frac{2d(\beta+d)}{2\beta+d} \end{aligned}$$

as is seen from Equation (2) and from

$$\begin{aligned} c(\kappa)^{-1} &= \int_{S_d}\left(1-(\kappa\delta(x,\eta))^\beta\right)_+ d\mu(x) \\ &= \mu_{d-1}(S_{d-1})\int_0^\pi \left(1-(\kappa\theta)^\beta\right)_+ \sin^{d-1}\theta\, d\theta \\ &= \mu_{d-1}(S_{d-1})\int_0^{\kappa\pi}\left(1-t^\beta\right)_+ \sin^{d-1}(\kappa^{-1}t)\kappa^{-1}dt \\ &\sim \kappa^{-d}\mu_{d-1}(S_{d-1})\int_0^1\left(1-t^\beta\right)t^{d-1}dt \\ &= \kappa^{-d}\mu_{d-1}(S_{d-1})\frac{\beta}{(\beta+d)d}. \end{aligned} \tag{5}$$

Let us denote $y = (1+\gamma)C_0 2\beta/(2\beta+d)$. Then it can be checked that

$$\frac{y^2}{D_\epsilon^2} = 2d(1+\gamma)^2\log(\epsilon^{-2/(2\beta+d)})$$

and for sufficiently small ϵ, because $Z_\epsilon(\eta(N,1)) \sim N(0,\psi_\epsilon^2 D_\epsilon^2)$,

$$\begin{aligned} P\left(\|\psi_\epsilon^{-1}Z_\epsilon\|_\infty > y\right) &\le P\left(\max_{i=1,\ldots,N}\psi_\epsilon^{-1}|Z_\epsilon(\eta(N,i))| > y\right) \\ &\le NP\left(D_\epsilon^{-1}\psi_\epsilon^{-1}|Z_\epsilon(\eta(N,1))| > \frac{y}{D_\epsilon}\right) \\ &\le N\exp\left\{-\frac{1}{2}\frac{y^2}{D_\epsilon^2}\right\} \\ &\le N\exp\left\{-\frac{d(1+\gamma)^2\log\epsilon^{-2}}{2\beta+d}\right\} \\ &\le \chi^{-1}(\log\epsilon^{-2})^{-d/(2\beta+d)}(\epsilon^2)^{d((1+\gamma)^2-1)/(2\beta+d)}, \end{aligned}$$

which approaches to 0 as $\epsilon \to 0$. $\square$

Next, let us consider the bias of the estimator.

Lemma 4

$$\limsup_{\epsilon\to 0} \sup_{f\in\Lambda(\beta,L)} \psi_\epsilon^{-1}\|f - E\hat{f}\|_\infty \le (1+\gamma)C_0\frac{d}{2\beta+d}.$$

Proof. We have that

$$\begin{aligned}
&\|E\hat{f} - f\|_\infty \\
&= \max_{i=1,\ldots,N} \sup_{x\in A_{Ni}} |E\hat{f}(x) - f(x)| \\
&\le \max_{i=1,\ldots,N} \sup_{x\in A_{Ni}} \left[|E\hat{f}(\eta(N,i)) - f(\eta(N,i))| + |f(x) - f(\eta(N,i))|\right] \\
&\le \max_{i=1,\ldots,N} \left[|E\hat{f}(\eta(N,i)) - f(\eta(N,i))| + L\sup_{x\in A_{Ni}} \delta(x,\eta(N,i))^\beta\right].
\end{aligned}$$

According to the comment after Lemma 5,

$$\limsup_{\epsilon\to 0} \sup_{x\in A_{Ni}} \delta(x,\eta(N,i)) \le \limsup_{\epsilon\to 0} \left(\frac{D}{N}\right)^{1/d} \tag{6}$$

for some constant $D>0$, and thus

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \psi_\epsilon^{-1} \sup_{x\in A_{Ni}} \delta(x,\eta(N,i))^\beta \\
&\le \limsup_{\epsilon\to 0} \psi_\epsilon^{-1}\left(\frac{D}{N}\right)^{\beta/d} \le (D\chi)^{\beta/d} \le \frac{\gamma C_0 d}{L(2\beta+d)}.
\end{aligned}$$

On the other hand, for $i=1,\ldots,N$,

$$\begin{aligned}
&|E\hat{f}(\eta(N,i)) - f(\eta(N,i))| \\
&= |E\hat{a}_{N,i} - f(\eta(N,i))| \\
&= \left|c(\kappa)\int_{S_d} K_{\kappa,\eta(N,i)}(x)f(x)d\mu(x) - f(\eta(N,i))\right| \\
&\le c(\kappa)\int_{S_d} K_{\kappa,\eta(N,i)}(x)\,|f(x) - f(\eta(N,i))|\,d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&\leq c(\kappa)L\int_{S_d}(1-(\kappa\delta(\eta(N,i),x))^\beta)_+\delta(\eta(N,i),x)^\beta d\mu(x)\\
&= c(\kappa)L\mu_{d-1}(S_{d-1})\int_0^\pi \left(1-(\kappa\theta)^\beta\right)_+\theta^\beta\sin^{d-1}\theta\,d\theta\\
&= c(\kappa)L\mu_{d-1}(S_{d-1})\int_0^{\kappa\pi}\left(1-t^\beta\right)_+(\kappa^{-1}t)^\beta\sin^{d-1}(\kappa^{-1}t)\kappa^{-1}dt\\
&\sim c(\kappa)L\kappa^{-\beta-d}\mu_{d-1}(S_{d-1})\int_0^1\left(1-t^\beta\right)t^{\beta+d-1}dt\\
&= c(\kappa)L\kappa^{-\beta-d}\mu_{d-1}(S_{d-1})\frac{\beta}{(2\beta+d)(\beta+d)}\\
&\sim \kappa^{-\beta}L\frac{d}{2\beta+d}=\psi_\epsilon C_0\frac{d}{2\beta+d}
\end{aligned}
$$

where we used Equation (5). □

The upper bound follows from lemma 3 and lemma 4 by an similar argument as was given in Korostelev (1993).

3 The Lower Bound

We will give the proof for the lower bound of Theorem 1. Let $J_{\kappa,\eta}: S_d\to\mathbf{R}$,

$$J_{\kappa,\eta}(x)=L\kappa^{-\beta}K_{\kappa,\eta}(x)=L\kappa^{-\beta}\left(1-(\kappa\delta(x,\eta))^\beta\right)_+$$

where $\kappa>0$, $\eta\in S_d$. Let $N=N(\kappa)$ be the greatest integer such that there exists such $\eta(N,i)\in S_d$, $i=1,\dots,N$, that the functions $J_{\kappa,\eta(N,i)}$ have disjoint supports. According to Lemma 6,

$$\liminf_{\kappa\to\infty}\kappa^{-d}N(\kappa)\geq Const. \tag{7}$$

Let

$$\mathcal{C}(\kappa,\{\eta(N,i)\})=\left\{\sum_{i=1}^N\theta_iJ_{\kappa,\eta(N,i)}:|\theta_i|\leq 1,\ i=1,\dots,N\right\}.$$

Now $\mathcal{C}(\kappa,\{\eta(N,i)\})\subset\Sigma(\beta,L)$. The complete class of estimators for estimating $f\in\mathcal{C}(\kappa,\{\eta(N,i)\})$ consists of all procedures of the form

$$\hat{f}_\epsilon=\sum_{i=1}^N\hat{\theta}_iJ_{\kappa,\eta(N,i)} \tag{8}$$

where $\hat{\theta}_i = \delta_i(y_1, \ldots, y_N)$, $i = 1, \ldots, N$, for measurable functions δ_i and

$$y_i = \int_{S_d} \xi_i(x) X(dx),\ i = 1, \ldots, N$$

with $\xi_i = J_{\kappa,\eta(N,i)}/\|J_{\kappa,\eta(N,i)}\|_2^2$. When $\hat{f}_\epsilon$ is of the form (8) and $f \in \mathcal{C}(\kappa, \{\eta(N,i)\})$ then

$$\begin{aligned} \left\|\hat{f}_\epsilon - f\right\|_\infty &\geq \max_{i=1,\ldots,N} \left|\hat{f}_\epsilon(\eta(N,i)) - f(\eta(N,i))\right| \\ &= \left|J_{\kappa,\eta(N,1)}(\eta(N,1))\right| \left\|\hat{\theta} - \theta\right\|_{l_N^\infty} \\ &= L\kappa^{-\beta} \left\|\hat{\theta} - \theta\right\|_{l_N^\infty}. \end{aligned}$$

Thus

$$\begin{aligned} r_\epsilon &\geq \inf_{\hat{f}_\epsilon} \sup_{f \in \mathcal{C}(\kappa, \{\eta(N,i)\})} E_{\epsilon,f} w\left(\psi_\epsilon^{-1} \left\|\hat{f}_\epsilon - f\right\|_\infty\right) \\ &\geq \inf_{\hat{\theta}} \sup_{|\theta_i| \leq 1} E_{N(\theta, \sigma_N^2 I_N)} w\left(\psi_\epsilon^{-1} L\kappa^{-\beta} \left\|\hat{\theta} - \theta\right\|_{l_N^\infty}\right) \end{aligned} \tag{9}$$

where $N(\theta, \sigma_n^2 I_N)$ is the normal distribution with the expectation vector θ and the covariance matrix $\sigma_N^2 I_N$, where I_N is the $N \times N$ identity matrix and $\sigma_N^2 = Var(y_1) = \epsilon^2/\|J_{\kappa,\eta(N,1)}\|_2^2$.

By Korostelev's lemma (Korostelev 1993, Donoho 1994), if

$$\sigma_N^{-1} \leq \sqrt{2-\gamma}\sqrt{\log N}, \tag{10}$$

for some $0 < \gamma < 2$, then

$$\inf_{\hat{\theta}} \sup_{|\theta_i| \leq 1} E_{N(\theta, \sigma_N^2 I_N)} w\left(\left\|\hat{\theta} - \theta\right\|_{l_N^\infty}\right) \longrightarrow w(1), \text{ when } N \longrightarrow \infty. \tag{11}$$

Let $0 < \gamma < 2$,

$$C_0' = L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{(2-\gamma)d}{2\beta+d}\right)^{\beta/(2\beta+d)},$$

and

$$\kappa = \left(\frac{C_0' \psi_\epsilon}{L}\right)^{-1/\beta} = \left(L^{d/\beta} A_{\beta,d}^{(2\beta+d)/\beta} \frac{(2-\gamma)d}{2\beta+d}\, \epsilon^2 \log \epsilon^{-2}\right)^{-1/(2\beta+d)}.$$

Now

$$
\begin{aligned}
\sigma_N^{-1} &= \epsilon^{-1} \left\| J_{\kappa,\eta(N,1)} \right\|_2 \\
&= \epsilon^{-1} \left\| L\kappa^{-\beta} K_{\kappa,\eta(N,1)} \right\|_2 \\
&= \epsilon^{-1} L\kappa^{-\beta-d/2} \left\| \kappa^{d/2} K_{\kappa,\eta(N,1)} \right\|_2 \\
&\sim \epsilon^{-1} L\kappa^{-\beta-d/2} A_{\beta,d}^{-(2\beta+d)/(2\beta)} \\
&= \left(\frac{(2-\gamma)d}{2\beta+d} \log \epsilon^{-2} \right)^{1/2} \\
&\leq (2-\gamma)^{1/2} \left(\log \left(\epsilon^2 \log \epsilon^{-2} \right)^{-d/(2\beta+d)} \right)^{1/2} \\
&= (2-\gamma)^{1/2} \sqrt{\log(Const. \times \kappa^d)}.
\end{aligned}
$$

Thus (10) is satisfied ultimately as $\epsilon \to 0$ by Equation (7).

The lower bound follows by (9) and (11) because

$$
\psi_\epsilon^{-1} L\kappa^{-\beta} = C_0' = L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{(2-\gamma)d}{2\beta+d} \right)^{\beta/(2\beta+d)}
$$

and $0 < \gamma < 2$ was chosen arbitrarily. □

4 Estimation with the L_2-loss

Let us prove Theorem 2. The proof of Theorem 2 follows from Pinsker (1980, Theorem 1). In fact, consider the observations

$$
y_{k,i} = \theta_{k,i} + \epsilon\xi_{k,i},\ k = 1, 2, \ldots,\ i = 1, \ldots, \tau_k,
$$

where $\tau_k \geq 1$ are integers, $\xi_{k,i}$ are i. i. d. $N(0,1)$, $\epsilon > 0$ is the noise size, and the sequence $\theta = (\theta_{k,i})$ is in l_2. Consider the parameter space

$$
\Theta = \left\{ \theta \in l_2 : \sum_{k=1}^{\infty} a_k \sum_{i=1}^{\tau_k} \theta_{k,i}^2 \leq M \right\}
$$

where $a_k \to \infty$ as $k \to \infty$. Define

$$R_\epsilon^*(\Theta) = \inf_{\hat\theta} \sup_{\theta \in \Theta} E_{\epsilon,\theta} \left\| \hat\theta - \theta \right\|_{l_2}^2 .$$

From Pinsker (1980, Theorem 1) it follows that

$$R_\epsilon^*(\Theta) \sim \omega^2 \quad (\epsilon \longrightarrow 0)$$

where

$$\omega^2 = \epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k \left(1 - (\mu_\epsilon a_k)^{1/2}\right) \tag{12}$$

and μ_ϵ is the unique solution of

$$\epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k a_k \left((\mu_\epsilon a_k)^{-1/2} - 1\right) = M. \tag{13}$$

If one chooses $a_k = \sigma_k^s$ and $\tau_k = \mathrm{dim} E_k^{(d)}$, where σ_k are the eigenvalues and $E_k^{(d)}$ are the eigenspaces of the Laplace-Beltrami operator, respectively, then it holds that

$$R_\epsilon(\Sigma(s, M)) \sim R_\epsilon^*(\Theta).$$

For the circle S_1 it is known that when $k \geq 1$, $\sigma_k = k^2$, $\mathrm{dim} E_k = 2$, and orthonormal basis of E_k is $\{\pi^{-1/2} \cos k\theta, \pi^{-1/2} \sin k\theta\}$. For the sphere S_2 it is known that $\sigma_k = k(k+1)$ and $\mathrm{dim} E_k = 2k+1$, when $k \geq 1$. For the d-dimensional sphere S_d it is known that $\sigma_k = k(k+d-1)$ and

$$\begin{aligned} \mathrm{dim} E_k^{(d)} &= \mathrm{card}\left\{(l_2, \ldots, l_d) \in \mathcal{Z}_+^{d-1} : k \geq l_2 \geq l_3 \geq \cdots \geq l_{d-1} \geq |l_d|\right\} \\ &= \frac{(2k+d-1)(k+d-2)!}{k!(d-1)!} \end{aligned}$$

(Takeuchi, 1994, pages 216, 223). It can be seen that

$$\mathrm{dim} E_k^{(d)} = \sum_{i=0}^{k} \mathrm{dim} E_i^{(d-1)} = \sum_{i=0}^{d-1} \alpha_i k^i,$$

where α_i are integers and $\alpha_{d-1} = 2/(d-1)!$.

Thus Theorem 2 follows by applying (12) if we choose $a_k = [k(k+d-1)]^s$ and $\tau_k = \sum_{i=0}^{d-1} \alpha_i k^i$, where α_i are integers and $\alpha_{d-1} = 2/(d-1)!$.

Appendix

We will prove two facts about spheres. Lemma 5 is needed to prove the upper bound for Theorem 1 and Lemma 6 is needed to prove the lower bound for Theorem 1.

Lemma 5 *Let $\eta(N,i) \in S_d$, $i = 1, \ldots, N$, be defined for sufficiently large N and be such that $\eta(N,i)$ are asymptotically equidistant. Let $\lambda = \lambda(N) > 0$ be the largest number such that $\cup_{i=1}^{N} B_{Ni}(\lambda) = S_d$, where*

$$B_{Ni}(\lambda) = \left\{ x \in S_d : \delta(x, \eta(N,i)) \le \lambda^{-1} \right\}. \tag{14}$$

There is such constant $0 < D < \infty$ that

$$\limsup_{N \to \infty} N \lambda(N)^{-d} \le D.$$

Proof. First note that

$$\mu(B_{Ni}(\lambda)) = \mu_{d-1}(S_{d-1}) \int_0^{\lambda^{-1}} \sin^{d-1} \theta d\theta \sim \mu_{d-1}(S_{d-1}) \lambda^{-d} \text{ as } \lambda \to \infty. \tag{15}$$

Let $\lambda' > 0$ be the smallest number such that $B_{Ni}(\lambda')$ are disjoint. Now

$$\lambda = Const \times \lambda'$$

where $Const > 1$. From Equation (15),

$$\mu(S_d) \ge \sum_{i=1}^{N} \mu\left(B_{Ni}(\lambda')\right) \sim N \mu_{d-1}(S_{d-1})(\lambda')^{-d}.$$

Thus, from the previous two displays,

$$\limsup_{N \to \infty} N \lambda^{-d} = Const \times \limsup_{N \to \infty} N(\lambda')^{-d} \le Const \times \frac{\mu(S_d)}{\mu_{d-1}(S_{d-1})}.$$

$\square$

Equation (6) follows from Lemma 5 because $A_{Ni} \subset B_{Ni}(\lambda)$ and thus

$$\sup_{x \in A_{Ni}} \delta(\eta(N,i), x) \le \lambda^{-1}$$

where A_{Ni} was defined before display (4).

Lemma 6 *For sufficiently large κ, let $N = N(\kappa)$ be such that $N \to \infty$ when $\kappa \to \infty$ and $\eta(N, i) \in S_d$, $i = 1, \dots, N$, are such that $\eta(N, i)$ are asymptotically equidistant, and such that $B_{Ni}(\kappa)$ are disjoint, when B_{Ni} is defined in (14). There is such constant $0 < D < \infty$ that*

$$\liminf_{\kappa \to \infty} N(\kappa)\kappa^{-d} \geq D.$$

Proof. Let $\kappa' > 0$ be the the largest number such that $\cup_{i=1}^{N(\kappa)} B_{N(\kappa)i}(\kappa') = S_d$. Now

$$\kappa' = Const \times \kappa$$

where $Const > 1$. From Equation (15),

$$\mu(S_d) \leq \sum_{i=1}^{N} \mu\left(B_{Ni}(\kappa')\right) \sim N\mu_{d-1}(S_{d-1})(\kappa')^{-d}.$$

Thus, from the previous two displays,

$$\liminf_{\kappa \to \infty} N\kappa^{-d} = Const \times \liminf_{\kappa \to \infty} N(\kappa')^{-d} \geq Const \times \frac{\mu(S_d)}{\mu_{d-1}(S_{d-1})}.$$

□

Acknowledgements

This research was supported by Sonderforschungsbereich 373 "Quantifikation und Simulation Ökonomischer Prozesse" and by Jenny and Antti Wihuri foundation.

References

[1] Giné, E. M. (1975). Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms. *Ann. Statist.* **3**, 1243–1266.

[2] Donoho, D. L. (1994). Asymptotic minimax risk for sup-norm loss: Solution via optimal recovery. *Probab. Theory Relat. Fields* **99**, 145–170

[3] Hall, P., Watson, G. S., and Cabrera, J. (1987). Kernel density estimation with spherical data. *Biometrika* **74**, 751–62.

[4] Hendriks, H. (1990). Nonparametric estimation of a probability density on a Riemannian manifold using Fourier expansions. *Ann. Statist.* **18**, 832–849.

[5] Korostelev, A. P. (1993). Exact asymptotically minimax estimator for nonparametric regression in uniform norm. *Theory Probab. Appl.* **38**, 775–782.

[6] Pinsker, M. S. (1980). Optimal filtering of square integrable signals in Gaussian white noise. *Problems Inform. Transmission* **16**, 52–68.

[7] Rudin, W. (1986). *Real and complex analysis.* 3rd ed. McGraw-Hill, Singapore.

[8] Takeuchi, M. (1994). *Modern spherical functions.* Translations of Mathematical Monographs Vol. 135. American Mathematical Society, Providence, Rhode Island.

Jussi Klemelä

Rolf Nevanlinna Institute

P.O. Box 4

SF-00014 UNIVERSITY OF HELSINKI

FINLAND