Level set trees and the analysis of shapes

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Abstract

A level set tree of a function is a tree structure of the separated components of the level sets of the function. The tree structure of local minima or maxima of a function can be described by a level set tree. Level set trees can be used to describe not only the shape of functions but also the shape of multidimensional sets; we can define a distance function or a height function on a set and construct a level set tree of this function. Level set trees can also be used to describe the shape of point clouds, by applying appropriate smoothing. With the help of a level set tree one can define shape isomorphic transforms. A shape isomorphic transform transforms a multidimensional object to a low-dimensional object which has same shape characteristics as the original multivariate object. This leads to a recursive analysis of the shape of a function: we start by analyzing the structure of local extremes of the function with level set trees and then continue to analyze the shape of the connected components of the level sets. A natural approach to mode testing consists of testing at each level whether the level set contains separated components. Level set trees provide a conceptual and computational framework for implementing such a testing procedure.

Key Words: Level set tree, shape isomorphism, visualization of multivariate functions.

1 Introduction

We are interested in the visualization of multivariate functions $f : \mathbf{R}^d \to \mathbf{R}$. We are particularly interested in the visualization of functions when $d > 3$. In order to visualize multivariate objects we have to transform these objects to 2- or 3-dimensional objects, since humans cannot see higher than 3-dimensional objects. Projections and slices are often applied to derive lower dimensional objects from high dimensional objects. We consider an other possibility: the use of *shape isomorphic transforms*. This approach uses the fact that it is possible to visualize a multidimensional object with a low dimensional object if these objects have the same shape. Usually a shape isomorphism is defined in topology in terms of homeomorphisms and diffeomorphisms. However, these definitions apply to objects of same dimension but we are interested in the similarity of objects of different dimensions.

A *volume transform* is the basic shape isomorphic transform which we discuss in this article. A volume transform is a transform which is used to visualize local extremes of a function. A volume transformed function has the same shape as the original function in the sense that the number and the sizes of the local extremes (either minima or maxima) are equal. We will use the term *mode isomorphism* to describe this type of similarity in shape.

To define a volume transform we use the concept of a level set tree. Let $f : \mathbf{M} \to \mathbf{R}$, where $\mathbf{M}$ is a Riemannian manifold. The main cases considered in this paper are $\mathbf{M} = \mathbf{R}^d$ and $\mathbf{M} = \mathbf{S}_{d-1}$, where $\mathbf{S}_{d-1}$ is the unit sphere: $\mathbf{S}_{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$. The (upper) level set of f with level $\lambda \in \mathbf{R}$ is defined by

$$\Lambda(f, \lambda) = \{x \in \mathbf{M} : f(x) \geq \lambda\}, \tag{1}$$

and the lower level set is defined by

$$\Lambda^-(f, \lambda) = \{x \in \mathbf{M} : f(x) \leq \lambda\}. \tag{2}$$

The exact level set is defined by

$$\Gamma(f, \lambda) = \{x \in \mathbf{M} : f(x) = \lambda\}. \tag{3}$$

The unqualified term "level set" will refer below to the upper level set.

Visualization of functions and sets are related to each other: a function can be visualized by visualizing a collection of its level sets, and a set can be visualized by defining a function on the set and then visualizing this function. We take as the basic tool the visualization of a function with a volume transform. We may proceed the visualization of a function by visualizing level sets of a function: define a function on the level sets and use a volume transform to visualize these level sets. This leads to a hierarchical exploration of the shape of a function. Under certain regularity conditions (level sets are star shaped, level sets of the boundary functions of level sets are star shaped, and so on), we get a cascade of simpler and simpler sets, and in a sense the complete shape of a function can be visualized.

Section 2 contains the basic definitions: the definitions of a level set tree and a volume transform. Section 3 elaborates on the concept of a volume transform by discussing which properties of a multivariate function remain invariant under a volume transform: mode isomorphisms are defined. Section 4 discusses how volume transforms can be used in a hierarchical exploration of the shape of a function.

2 Level set trees and volume functions

We define a volume function as a discrete structure in Section 2.1. This definition is sufficient for practical purposes. In Section 2.2 we define a limit volume function, which may be used to clarify general properties of a volume function.

2.1 Volume transform

We use upper level sets to define a transform of a multivariate function to a univariate function. This *volume transform* can be used to visualize the local maxima of a function: we visualize the number, size, and the tree structure of the local maxima. We may apply lower level sets to define an analogous transform to visualize the local minima of a function. Combined together, these transforms give a comprehensive visualization of the local extremes of a multivariate function. The concept of a level set tree is the basic concept underlying the definition of a volume function. A level set tree and a volume function were defined in [1] for piecewise constant functions. Here we introduce more general definitions.

Definition 1 (Level set tree.) *Let* $\mathbf{M}$ *be a Riemannian manifold. A level set tree of function* $f : \mathbf{M} \to \mathbf{R}$*, associated with set of levels* $\mathcal{L} = \{\lambda_1 < \cdots < \lambda_L\}$*, where* $\lambda_L \le \sup_{x \in \mathbf{M}} f(x)$*, is a tree whose nodes are associated with subsets of* $\mathbf{M}$ *and levels in* $\mathcal{L}$ *in the following way.*

1. *Write*
$$\Lambda(f, \lambda_1) = A_1 \cup \cdots \cup A_K,$$
where sets A_i *are pairwise separated, and each is connected. The level set tree has* K *root nodes which are associated with sets* A_i*,* $i = 1, \ldots, K$*, and each root node is associated with the same level* λ_1*.*

2. *Let node m be associated with set $B \subset \mathbf{M}$ and level $\lambda_l \in \mathcal{L}$, $1 \leq l < L$.*

   (a) *If $B \cap \Lambda(f, \lambda_{l+1}) = \emptyset$, then node m is a leaf node.*

   (b) *Otherwise, write*

   $$B \cap \Lambda(f, \lambda_{l+1}) = C_1 \cup \cdots \cup C_M,$$

   where sets C_i are pairwise separated, and each is connected. Then node m has M children, which are associated with sets C_i, $i = 1, \ldots, M$, and each child is associated with the same level λ_{l+1}.

Above we say that sets $B, C \subset \mathbf{M}$ are *separated* if $\inf\{\delta(x, y) : x \in B,\ y \in C\} > 0$, where $\delta(x, y)$ denotes the Riemannian distance and we say that set $A \subset \mathbf{M}$ is *connected* if for each nonempty $B, C \subset \mathbf{M}$ such that $A = B \cup C$, sets B and C are not separated. Thus, two sets are separated if there is some space between them and a set is connected if it cannot be written as a union of two separated sets. Now we are ready to define a volume function. A volume transform is defined as the mapping which maps a function to its volume function.

Definition 2 (Volume function.) *Let $f : \mathbf{M} \to \mathbf{R}$ be a function, let μ be a Borel measure on $\mathbf{M}$, and let T be a level set tree of f.*

- *Annotate each node m of the level set tree T with an interval $\mathcal{I}_m \subset \mathbf{R}$. Let the length of an interval be equal to the μ-volume of the set annotated with the node. Let the intervals be nested according to the tree structure of the level set tree. Remark 1 comments on the exact definition of the intervals.*

- *volume function $v(f; T) : \mathbf{R} \to \mathbf{R}$ is such that for each level $\lambda \in \mathbf{R}$,*

  $$\{x \in \mathbf{R} : v(f)(x) \geq \lambda\} =$$
  $$\bigcup \{\mathcal{I}_m : m \text{ is such node of } T \text{ that } \lambda_m \geq \lambda\},$$

  where λ_m is the level and $\mathcal{I}_m$ is the interval associated to node m.

There exists two approaches to define a level set tree.

1. We may define a level set tree only for piecewise constant functions. If a function is not piecewise constant we may first approximate it with a piecewise constant function, and then calculate the level set tree of the piecewise constant approximation. This approach was used in [1].

We say that a function $f : \mathbf{M} \to \mathbf{R}$ is piecewise constant if its range is a finite set:

$$\text{range}(f) \stackrel{def}{=} \{f(x) : x \in \mathbf{M}\} = \{\lambda_1, \ldots, \lambda_N\} \tag{4}$$

where $\lambda_1 < \cdots < \lambda_N$.

2. In this article we have defined level set trees for a larger class of functions, and the discretization is made by choosing a grid of levels. This approach is more general because Definition 1 applies also to piecewise constant functions. In particular, if the function is piecewise constant, as defined in (4), then we may choose the grid of levels of the level set tree to be $\lambda_1, \ldots, \lambda_N$.

Definition 3 *If a function is piecewise constant as defined in (4), then the level set tree with the grid of levels* $\lambda_1, \ldots, \lambda_N$ *is called the* saturated level set tree *of the piecewise constant function.*

The saturated level set tree of a piecewise constant function is such that adding more levels to the grid does not lead to a more accurate representation of the function with its level set tree.

A drawback of Definition 1 is that it is difficult to formulate general regularity conditions which would guarantee the applicability of the definition, but at the same time would not be overly restrictive. Remark 2 discusses regularity conditions which are needed in Definition 1.

Remark 1 Definition 2 does not specify the locations of the intervals associated to the nodes of a level set tree. We could use rather arbitrary rules, but the following rule is quite natural. Choose first an interval $[0, L]$, where L is greater than the sum of the volumes of the sets associated to the root nodes. Then the intervals associated to the root nodes are positioned inside $[0, L]$ in a symmetric way. After that, one positions the intervals recursively, making a nested collection of intervals according to the tree structure, and positioning the intervals symmetrically. Note that we have note excluded the case where some of the level sets of the function have infinite volume. Note also that we have not defined a level set tree as an ordered tree, so that the positioning of the sibling intervals may be done in an arbitrary order.

Remark 2 A level set tree is not defined for every function, since it may not be possible to decompose each level set of a function to a finite number of separated components. In practice the function we want to visualize may often be approximated by a function which is piecewise constant on some simple sets, like rectangles. For these kind of functions a level set tree is always defined. Also, for Morse functions a level set tree is always defined. A function $f : \mathbf{M} \to \mathbf{R}$ is a Morse function if it is a smooth function (infinitely differentiable) and all its critical points are nondegenerate. Point $x \in \mathbf{M}$ is a *critical point* of f if all the partial derivatives of f vanish at x. A critical point x is *nondegenerate* if $\det(d_x^2 f) \neq 0$, where $d_x^2 f$ is the Hessian matrix of the second partial derivatives of f at x.

Remark 3 Level set trees contain certain ambiguity because the grid of levels of a level set tree is arbitrary, and not determined by the underlying function. We can remove to a certain extent the ambiguity in a volume function by defining a limit volume function, see Definition 4 below.

2.2 Limit volume function

Definition 2 of a volume function depends on a level set tree, as defined in Definition 1, which depends in turn on a finite grid of levels, which were used to construct this level set tree. It is of interest to define a limit volume function which would be independent of any grid of levels. A natural way to define a limit function would be to choose a sequence of piecewise constant functions f_k which converge to f as $k \to \infty$ (in $L_1(\mathbf{M})$, for example), calculate a volume function $v(f_k; T_k)$ for each k, and define a limit volume function $v(f)$ as the limit of $v(f_k; T_k)$ as $k \to \infty$ (in $L_1(\mathbf{R})$, for example). A naive version of this approach does not work since the ordering of the nodes in the level set trees affects the volume function, and we want to make the definition independent of this ordering. Thus we are led to the following definition.

Definition 4 (Limit volume function.) *Let $f : \mathbf{M} \to \mathbf{R}$ be a bounded function. Function $v(f) : \mathbf{R} \to \mathbf{R}$ is a* limit volume function *of f, if for each sequence of grids of levels $\mathcal{L}_k = \{\lambda_{k,1} < \cdots < \lambda_{k,N_k}\}$, $k = 1, 2, \ldots$, such that*

1) $\lambda_{k,N_k} \leq \sup_{x\in\mathbf{M}} f(x)$, and

2) $f_k \to f$ in $L_1(\mathbf{M})$, where

$$f_k(x) = \sum_{i=1}^{N_k} \lambda_{k,i} I_{A_{k,i}}(x), \qquad x \in \mathbf{M}, \tag{5}$$

with

$$A_{k,i} = \Lambda(f, \lambda_{k,i}) \setminus \Lambda(f, \lambda_{k,i+1}),$$

$i = 1, \ldots, N_k - 1$,

$$A_{k,N_k} = \Lambda(f, \lambda_{k,N_k}),$$

we can find a sequence of level set trees T_k *and volume functions* $v(f_k; T_k)$ *such that*

a) $\mathcal{L}_k$ *is the grid of levels of level set tree* T_k*, and*

b) $v(f_k; T_k)$ *converges to* $v(f)$ *in* $L_1(\mathbf{R})$*, as* $k \to \infty$*.*

3 Mode isomorphisms

We justify the usefulness of a volume function with the concept of a *mode isomorphism*: a volume function is useful because it is a 1D function which is mode isomorphic to the original multivariate function.

To define a mode isomorphism we need first to define the excess mass associated with a node of a level set tree and second we need to define the excess mass isomorphism of level set trees.

For piecewise constant functions an excess mass associated with a node of a level set tree is equal to

$$\int_A (f - \lambda)\, d\mu, \tag{6}$$

where A is the set associated with the node and λ is the level associated with the node, see (8) for a precise expression. For general functions the excess mass associated with a node of a level set tree will typically converge to the integral in (6), when the grid of levels of the level set tree becomes finer.

To define the excess mass associated with a node of a level set tree we use the following notations. Assume that with node m of a level set tree there is associated level λ and set A. Then we write

$$\mathrm{set}(m) = A, \qquad \mathrm{level}(m) = \lambda.$$

Furthermore, with $\mathrm{parent}(m)$ we mean the unique parent of node m. We say that a node is a descendant of node m if it is either a child of m or a child of an other descendant of m.

Definition 5 The excess mass associated with node m *of a level set tree is defined by*

$$\begin{aligned} \mathrm{excmass}(m) \;\; = \;\; & \sum \{\mu(\mathrm{set}(\mathrm{parent}(m_0))) \\ & \times [\mathrm{level}(m_0) - \mathrm{level}(\mathrm{parent}(m_0))] : \\ & m_0 \textit{ is a descendant of } m\}\,. \end{aligned} \tag{7}$$

Remark 4 For piecewise constant functions the sum in the right hand side of (7) may be written as an integral. We assume that the level set tree T is saturated, that is, the grid of the levels of the level set tree is $\mathcal{L} = \text{range}(f)$. Then, for every node m of T,

$$\text{excmass}(m) = \int_{\text{set}(m)} (f(x) - \text{level}(m))\, d\mu(x). \tag{8}$$

In words, the excess mass is the volume of the area which the piecewise constant function delineates over a given level, in a given branch of the level set tree.

Level set trees are said to be excess mass isomorphic if the level set trees have isomorphic tree structures and if the excess masses of the corresponding nodes are equal. We say that trees T_1 and T_2 are isomorphic, when there is a bijection I from the set of nodes of T_1 to the set of nodes of T_2, such that if m_0 and m_1 are nodes of T_1 and m_0 is the parent of m_1, then $I(m_0)$ is the parent of $I(m_1)$.

Definition 6 *Level set trees T_1 and T_2 are* excess mass isomorphic *when*

1. *trees T_1 and T_2 are isomorphic,*
2. *for every node m of T_1,*

$$\text{excmass}(m) = \text{excmass}(I(m))$$

where I is the isomorphism between T_1 and T_2.

Now we are ready to define the concept of a mode isomorphism. Mode isomorphism is defined between functions which are defined on possibly different manifolds $\mathbf{M}_1$ and $\mathbf{M}_2$. For example, the manifolds could be Euclidean spaces with different dimensions: $\mathbf{M}_1 = \mathbf{R}^d$ and $\mathbf{M}_2 = \mathbf{R}^{d'}$.

Definition 7 (Mode isomorphism.) *Let $f : \mathbf{M}_1 \to \mathbf{R}$ and $g : \mathbf{M}_2 \to \mathbf{R}$ be bounded functions. Denote*

$$\lambda_0^{(f)} = \sup_{x \in \mathbf{M}_1} f(x), \qquad \lambda_0^{(g)} = \sup_{x \in \mathbf{M}_2} g(x),$$

and

$$\Delta_{f,g} = \lambda_0^{(g)} - \lambda_0^{(f)}. \tag{9}$$

Functions f and g are mode isomorphic*, when for all*

$$\lambda_1 < \cdots < \lambda_N \le \sup_{x \in \mathbf{M}_1} f(x),$$

level set trees T_f of f and T_g of g are excess mass isomorphic, where T_f has grid of levels $\{\lambda_1, \ldots, \lambda_N\}$, and T_g has grid of levels $\{\lambda_1 + \Delta_{f,g}, \ldots, \lambda_N + \Delta_{f,g}\}$.

We list some properties of a mode isomorphism. Let f and g be mode isomorphic. Let $\Delta_{f,g}$ be defined in (9). Then,

1. f and g have the same number of local maxima,
2. the level set of f with level λ has the same number of separated components as the level set of g with level $\lambda + \Delta_{f,g}$,
3. for all $\lambda \in \mathbf{R}$,
$$\int_{(f \geq \lambda)} f \, d\mu_1 = \int_{(g \geq \lambda + \Delta_{f,g})} g \, d\mu_2,$$
where we use the notation $(f \geq \lambda) = \{x \in \mathbf{M}_1 : f(x) \geq \lambda\}$.

We can also tie the concept of a mode isomorphism and the concept of a limit volume function together.

Proposition 1 *Let $f : \mathbf{M} \to \mathbf{R}$. Function f and a limit volume function $v(f)$ of f, defined in Definition 4, are mode isomorphic, if $v(f)$ is such that $\sup_{x \in \mathbf{M}} f(x) = \sup_{t \in \mathbf{R}} v(f)(t)$.*

Proof. If f and $v(f)$ are not mode isomorphic, then there exists a grid $\mathcal{L}$ of levels such that the corresponding level set trees are not excess mass isomorphic. This implies that for some $\lambda \in \mathcal{L}$,
$$0 < \left| \int_{\mathbf{M}} (f - \lambda)_+ - \int_{-\infty}^{\infty} (v(f) - \lambda)_+ \right|. \tag{10}$$
Let $\mathcal{L}_k$ be such sequence of grids of levels that $\lambda \in \mathcal{L}_k$, for $k \geq k_0$, for some k_0, and satisfying the conditions 1) and 2) of Definition 4. Let T_k be the level set tree of f corresponding to grid $\mathcal{L}_k$, and let f_k be the quantization (5) of f corresponding to grid $\mathcal{L}_k$. We have
$$\begin{aligned}
&\left| \int_{\mathbf{M}} (f - \lambda)_+ - \int_{-\infty}^{\infty} (v(f) - \lambda)_+ \right| \\
\leq\ & \left| \int_{\mathbf{M}} (f - \lambda)_+ - \int_{\mathbf{M}} (f_k - \lambda)_+ \right| \\
&+ \left| \int_{\mathbf{M}} (f_k - \lambda)_+ - \int_{-\infty}^{\infty} (v(f; T_k) - \lambda)_+ \right| \\
&+ \left| \int_{-\infty}^{\infty} (v(f; T_k) - \lambda)_+ - \int_{-\infty}^{\infty} (v(f) - \lambda)_+ \right| \\
\stackrel{def}{=}\ & A_{1,k} + A_{2,k} + A_{3,k}.
\end{aligned}$$
We have that $\lim_{k \to \infty} A_{1,k} = 0$, since $f_k \to f$ in L_1, by the choice of $\mathcal{L}_k$. Also, $A_{2,k} = 0$ for all $k \geq k_0$, since by the construction, f_k and $v(f; T_k)$ are mode

isomorphic. (Indeed, the level set trees of f_k and $v(f; T_k)$ are isomorphic by the construction. These level set trees are also excess mass isomorphic because the level sets of f_k and $v(f; T_k)$ have the same volumes and the same levels, and the definition of excess mass in (7) depends only on the volumes and levels of the level sets.) Finally, $\lim_{k\to\infty} A_{3,k} = 0$, since $v(f; T_k) \to v(f)$ in L_1, as $k \to \infty$, by the definition of a limit volume function in Definition 4. We have reached a contradiction with (10). QED

4 Functions defined on a set and a cascade of a function

A volume function is a univariate function and thus it contains only a small amount of information about the original multivariate function. We can reach a more detailed visualization when we continue to visualize the connected components of the level sets of the function. Indeed, the nodes of a level set tree are associated with connected components of manifold $\mathbf{M}$, and we should visualize those components.

Thus we have to discuss methods for visualizing connected subsets of manifold $\mathbf{M}$, whereas previously we discussed methods for visualizing functions $f : \mathbf{M} \to \mathbf{R}$. However, we approach the problem of set visualization by defining a function on this set; we transform the task of set visualization to the task of function visualization, and then we can use the previous tools (volume function). We start with the case $\mathbf{M} = \mathbf{R}^k$. A function on a set $A \subset \mathbf{R}^k$ may be defined at least in the following ways.

1. A *distance function* $f_A : A \to \mathbf{R}$ of $A \subset \mathbf{R}^k$ is defined by $f_A(x) = \|m - x\|$, where $m \in \mathbf{R}^k$. Thus a distance function is constructed by foliating the Euclidean space into the union of cocentric spheres.

2. A *height function* $f_A : A \to \mathbf{R}$ of $A \subset \mathbf{R}^k$ is the orthogonal projection of A onto a fixed line on $\mathbf{R}^k$, when the line is identified with $\mathbf{R}$. Thus a height function is constructed by foliating the Euclidean space into a union of parallel planes.

A third kind of a function on a set is the boundary function of a star shaped set. A distance function and a height function are defined on any set, but the boundary function will be defined only for star shaped sets. We define star shaped sets first for $A \subset \mathbf{R}^d$ and then for $A \subset \mathbf{S}_{d-1}$, where $\mathbf{S}_{d-1}$ is the unit sphere: $\mathbf{S}_{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$. (1) We say that set $A \subset \mathbf{R}^d$ is *star shaped* if there is a reference point $m \in A$ and a *boundary function*

$f_A : \mathbf{S}_{d-1} \to [0, \infty)$ so that we can write the set as

$$A = \{m + r\xi : \xi \in \mathbf{S}_{d-1},\ \ 0 \le r \le f_A(\xi)\}.$$

(2) We say that set $A \subset \mathbf{S}_{d-1}$, $d \ge 2$, is *star shaped* if there is a reference point $m \in A$ and a *boundary function* $f_A : T_\eta \to [0, \pi]$, $T_\eta = \{\xi \in \mathbf{S}_{d-1} : \xi \perp \eta\}$, so that we can write the set as

$$A = \{m \cdot \cos\theta + \xi \cdot \sin\theta : \xi \in T_\eta,\ \ 0 \le \theta \le f_A(\xi)\}.$$

Note that set T_η is isomorphic with $\mathbf{S}_{d-2}$. A distance function and a boundary function are related to each other: both describe how the set is extending in various directions around a center point m. A natural choice for the center point is the barycenter: $m = \int_A x\, d\mu(x) / \int_A\, d\mu$. The barycenter is the center of the mass of the set. A height function is based on a one dimensional projection and thus one would need several height functions to visualize all tails of a star shaped set. When the set is star shaped, then it may be reasonable to use a boundary function instead of a distance function: the boundary function is defined on the unit sphere, which is a lower dimensional manifold than than the Euclidean space where the set is defined. Thus we achieve a dimension reduction when we describe a star shaped set with its boundary function.

One may now consider a recursive scheme for the visualization of a function. We start by constructing a level set tree of the function, and draw the volume function. Then we define a set function (a height function, distance function, or boundary function) for the connected components associated with the nodes of the level set tree. We construct a level set tree for each set function. One continues in this way, defining at each step new set functions for the connected components of the sets associated with the nodes of the level set trees of the previous step. However, the recursive scheme seems most natural when one uses boundary functions, instead of height or distance functions. This is due to the fact that the dimension of the boundary functions will decrease at each step, and these functions will reflect better the inherent shape of the sets (assuming that the sets are star shaped).

We now describe in more detail the scheme for constructing a nested structure of level set trees which zoom into the shape of the function, when boundary functions are used to represent star shaped sets. The fundamental idea is that if a k-dimensional set is star shaped, then it can be represented with a $k-1$-dimensional boundary function. Starting with a function $\mathbf{R}^d \to \mathbf{R}$, we construct a level set tree of the function and make the volume function. Then we take a connected component of a level set of f under closer inspection. If this d-dimensional set is star shaped, we can find its

$d-1$-dimensional boundary function, construct a level set tree and a volume function of the boundary function. We may continue in this way, taking next a connected component of a level set of the boundary function under closer inspection. At each step the dimension is decreasing by one, so that after d steps we end up with a one-dimensional function. We define a cascade of a function to be a tree structure of level set trees.

Definition 8 *A* Cascade of a function $f : \mathbf{R}^d \to \mathbf{R}$ *is a tree whose nodes are annotated with level set trees. The cascade has d levels.*

- *Level 1: the root node of the cascade is annotated with a level set tree of f.*
- *Level 2: let $\mathcal{C}$ be the collection of those sets $A \subset \mathbf{R}^d$, associated with the nodes of the level set tree of f, which are star shaped. Construct a boundary function $g : \mathbf{S}_{d-1} \to \mathbf{R}$ for each set $A \in \mathcal{C}$ and construct a level set tree for each function g. These level set trees are the children of the root node.*
- *Levels $k \in \{3, \ldots, d\}$: let us consider a node n of the cascade at level $k-1$, annotated with a level set tree of a function $g : \mathbf{S}_{d-k+2} \to \mathbf{R}$. Let $\mathcal{C}$ be the collection of those sets $A \subset \mathbf{S}_{d-k+2}$, associated with the nodes of the level set tree of g, which are star shaped. Construct a boundary function $g : \mathbf{S}_{d-k+1} \to \mathbf{R}$ for each set $A \in \mathcal{C}$ and construct a level tree for each function g. These level set trees are the children of n.*

Remark 5 Note that the level set tree of a distance function of a set is the same as a shape tree of the set, when a shape tree is defined in [2].

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