Lecture Notes 2

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1 Absolute Pricing

Let us consider a coin-tossing game where a participant receives one Euro when heads occur and zero Euros when tails occurs. The probability of getting heads is 1/2 and the probability of obtaining tails is 1/2. What is the fair price to participate in this game? It can be argued that the fair price is the expected gain:

$$0.5 \cdot 1 \text{ Euros} + 0.5 \cdot 0 \text{ Euros} = 0.5 \text{ Euros}.$$

The fairness of the price can be justified by the law of large numbers. The law of large numbers implies that the gain from repeated independent repetitions of the game with price 0.5 Euros converges to zero with probability one. A larger price than 0.5 Euros would give an almost sure profit to the organizer of the game in the long run and a smaller price than 0.5 Euros would give an almost sure profit to the player of the game in the long run.

It does not seem as clear what the price should be if we change the game so that a participant receives one million Euros when heads occur and zero Euros when tails occurs. Only few people would be willing to invest half a million Euros in order to participate in this game. The law of large numbers cannot be applied to justify a price because the probability of a bankruptcy is quite large when a player repeats the game.[1]

It can be argued that the price of the game should be equal to the expected utility: Let S be the random variable with $P(S = u) = 0.5$ and $P(S = 0) = 0.5$, where $u = 1$ million Euros. Then the expected utility is $Eu(S)$, where $u : [0, \infty) \to \mathbf{R}$ is a utility function.

[1] Note also that a doubling strategy gives an almost sure win. A player who follows the classical doubling strategy doubles his bet until the first time he wins. If he starts with 1 Euro, his final gain is 1 Euro almost surely.

The St. Petersburg paradox can be used to argue that a utility function should be used. In the St. Petersburg paradox the banker flips the coin until the heads come out the first time. The player receives 2^{k-1} coins when there are k tosses of the coin (1 coin if the heads come out in the first toss, 2 coins if the heads come out in the second toss, 4 coins if the heads come out in the third toss, and so on). What is the fair entrance fee to the game? We can calculate the expected gain. The probability that there are k tosses is $p_k = 2^{-k}$. Thus the expected payoff is

$$\sum_{k=1}^{\infty} p_k 2^{k-1} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

Thus it would seem that the entrance fee could be arbitrarily high. However, applying common sense, it does not seem reasonable to pay a high entrance fee. The paradox can be solved by using a utility function to measure the utility of the wealth. For example, the logarithmic utility function $x \mapsto \log_e(x)$ gives the expected utility of the game

$$\sum_{k=1}^{\infty} p_k \log_e(2^{k-1}) = \log_e 2,$$

which would give the price of 2 coins for the game.

Utility functions are rarely used in derivative pricing. We can assume that we are close to the setting of coin flipping games with small bets relative to the total wealth of the participants, so that utility functions do not play a big role. (In the neighborhood of one we can approximate the logarithmic function with a linear function.) Also, derivative prices have to be consistent with the prices of the basic assets, so that the principles of relative pricing are important.

2 Relative Pricing with Arbitrage

We illustrate relative pricing with arbitrage using a coin tossing example. After that, arbitrage and the law of one price are discussed more generally.

2.1 Pricing in a One Period Binary Model

Let us consider two games related to the same tossing of a coin. The first game is such that the player receives u Euros when heads occur and d Euros when tails occurs, where $u > d \geq 0$. The participation to this game is an analogy to buying a stock and we denote with S the random variable with $P(S = u) = 0.5$ and $P(S = d) = 0.5$.

The second game is such that the player receives one Euros when heads occur and zero Euros when tails occurs. The participation to this game is an analogy to buying a derivative. The second game can be considered as a derivative because the payoff of the second game is a random variable $H = f(S)$ for $f : \{u, d\} \to \mathbf{R}$, where $f(u) = 1$ and $f(d) = 0$. Random variable H has the distribution $P(H = 1) = 0.5$ and $P(H = 0) = 0.5$. The third asset is a bond with value $B = 1$. The price of bond is 1 and the price of stock is denoted with $\pi(S)$. We want to find the price of the derivative.

The derivative can be replicated with the bond and the stock: Consider the portfolio with ξ_1 bonds and ξ_2 stocks. We choose

$$\xi_1 = \frac{-d}{u-d}, \qquad \xi_2 = \frac{1}{u-d}.$$

The portfolio is $V = \xi_1 B + \xi_2 S$ and $P(V = H) = 1$, because

$$\begin{aligned} \xi_1 + \xi_2 d &= 0, \\ \xi_1 + \xi_2 u &= 1. \end{aligned}$$

By the law of one price, to exclude the possibility of arbitrage, the price of the derivative has to be equal with the price of the portfolio:[2]

$$\pi(H) = \pi(V).$$

The price of the portfolio is

$$\pi(V) = \xi_1 + \xi_2 \pi(S).$$

Thus the price of the derivative is

$$\pi(H) = \frac{\pi(S) - d}{u - d}. \tag{1}$$

The price of the derivative is in general not equal to 0.5. If $\pi(S) = (u+d)/2$, then the price of the derivative is $\pi(H) = 0.5$. If $\pi(S) < (u+d)/2$, then the price of the derivative satisfies $\pi(H) < 0.5$.

We have given the price of the derivative in (1) in terms of the price of the stock. This is an example of relative pricing: a price of an asset is given in terms of the prices of the other assets.

[2] If $\pi(H) < \pi(V)$, then buying H and selling V would give an almost sure profit. If $\pi(H) > \pi(V)$, then selling H and buying V would give an almost sure profit.

2.2 Arbitrage and the Law of One Price

Arbitrage is a term used in common language and a technical term used in mathematical finance. The absence of arbitrage implies the law of one price.

2.2.1 Arbitrage

Arbitrage is used in everyday language to denote a financial operation where one obtains a profit with probability one by a simultaneous selling and buying of assets. We give two examples of this type of arbitrage.

1. The stock of Daimler is listed both in Frankfurt and Stuttgart stock exchanges. If the stock can be bought in Frankfurt with the price of 10 Euros and sold in Stuttgart with the price of 11 Euros, we obtain a risk free profit of one Euro (minus the transaction costs).

2. Suppose the price of a stock is 10 Euros and a call option with strike price $K = 8$ Euros with the expiration time in one week can be bought with the price of 1 Euro. Then we can sell the stock short and buy the call option. The profit of the operation will be $-1 + 10 - 8 = 1$ Euro (buying the call costs 1 Euro, selling the stock short gives 10 Euros, and exercising the option costs 8 Euros).

   In general, we have a lower bound $S_t - K$ for the price of a call option, where S_t is the price of the stock at the time of buying the option, and K is the strike price.

In mathematical finance an arbitrage is a financial operation whose payoff is always non-negative and sometimes positive, that is, the probability of a non-negative payoff is one and the probability of a positive payoff is greater than zero. A reasonable system of prices should be such that arbitrage is excluded.

2.2.2 The Law of One Price

The law of one price states that if two financial instruments have the same payoffs then they have the same price. The absence of arbitrage implies that the law of one price holds. Indeed, consider the case where the law of one price does not hold. Then we have two assets with different prices at time zero, say $V_0^{(a)} < V_0^{(b)}$, and the values of the assets are the same with probability one at a later time: $P_0(V_1^{(a)} = V_1^{(b)}) = 1$. Then we can by the cheaper asset at time zero and sell the more expensive asset at time zero to obtain the amount $V_0^{(b)} - V_0^{(a)} > 0$. This amount can be put into a bank account. At time one the two assets have the same price, and thus we have locked the profit of time zero. We have shown that there exists an arbitrage opportunity. Thus we have shown that the absence of arbitrage implies that the law of one price holds.

2.3 Pricing using The Law of One Price

The law of one price can be used to price linear assets by replication.[3] Furthermore, the law of one price can be used to price all assets in complete markets. By a market we mean a collections of tradable assets together with assumptions about the probability distributions of the asset values. A complete market is such that any possible payoff can be obtained by a portfolio of assets. That is, assume that the market has tradable assets $S_t^1, \ldots, S_t^N$. Assume that an arbitrary payoff H_T can be obtained, so that $P_t(\xi^1 S_T^1 + \cdots + \xi^N S_T^N = H_T) = 1$. The law of one price implies that price of this payoff is $\pi(H_T) = \xi^1 S_t^1 + \cdots + \xi^N S_t^N$.

Futures are linear derivatives, and thus the law of one price can be used to price futures. Futures can be priced by the law of one price because futures can be defined as a portfolio of the underlying asset and a bond: the payoff of a futures contract is a linear combination of the payoffs of the underlying asset and a bond.

The payoff of an option is not linear function of the payoff of the underlying an thus options cannot be priced as easily as futures. The Black-Scholes model is a complete model for the markets, and thus the law of one price can be used to price options in the Black-Scholes model.

The no-arbitrage principle can be used to give bounds to option prices without assuming the Black-Scholes model, or any other restrictive market model. The no-arbitrage principle can also be used to show that the prices of two options should be the same, as in the case of the put-call parity.

3 Relative Pricing with Statistical Arbitrage

We have derived the price of the derivative in (1) using the replication of the derivative with a stock and a bond. The exact replication is possible only under special circumstances. It suffices to move from the binary model to a ternary model to make exact replication impossible so that only approximate replication is possible.

[3] We can use the arbitrage argument directly, but we have noted that the absence of arbitrage implies the law of one price and thus we use below the pricing with the replication.

3.1 Pricing in a One Period Ternary Model

Let us have two games related to the same tossing of a dice. The first game is such that the player receives d euros when the dice shows 1 or 2, c euros when the dice shows 3 or 4, and u euros when the dice shows 5 or 6, where $0 \le d < c < u$. The participation to this game is an analogy to buying a stock and we denote with S the random variable with $P(S = d) = P(S = c) = P(S = u) = 1/3$.

The second game is such that the player receives zero euros when the dice shows 1, 2, 3, or 4 and one euro when the dice shows 5 or 6. The participation to this game is an analogy to buying a derivative and we denote with H the random variable $H = f(S)$, where $f : \{1, \ldots, 6\} \to \mathbf{R}$ is defined by $f(x) = 0$ when $x \in \{1, \ldots, 4\}$ and $f(x) = 1$ when $x \in \{5, 6\}$. Now $P(H = 0) = 2/3$ and $P(H = 1) = 1/3$. The third asset is a bond with value $B = 1$. The price of the bond is 1 and the price of the stock is denoted with $\pi(S)$. We want to find the price $\pi(H)$ of the derivative.

The derivative cannot be replicated with the bond and the stock: Consider the portfolio with ξ_1 bonds and ξ_2 stocks. The portfolio is $V = \xi_1 B + \xi_2 S$. We have $P(V = H) = 1$ when ξ_1 and ξ_2 satisfy

$$\begin{aligned} \xi_1 + \xi_2 d &= 0, \\ \xi_1 + \xi_2 c &= 0, \\ \xi_1 + \xi_2 u &= 1. \end{aligned}$$

We can typically not find such ξ_1 and ξ_2 because, in general, two parameters cannot satisfy three equations simultaneously. To obtain an approximate replication we could choose ξ_1 and ξ_2 so that $E(V - H)^2$ is minimized. We have that

$$\begin{aligned} & E(V - H)^2 \\ &= P(S = d)(\xi_1 + \xi_2 d)^2 + P(S = c)(\xi_1 + \xi_2 c)^2 + P(S = u)(\xi_1 + \xi_2 u - 1)^2. \end{aligned}$$

Since the probabilities are all equal to 1/3, we get the least squares solution for $\xi = (\xi_1, \xi_2)'$:

$$\xi = (\mathcal{X}'\mathcal{X})^{-1}\mathcal{X}'\mathcal{Y},$$

where

$$\mathcal{X} = \begin{bmatrix} 1 & d \\ 1 & c \\ 1 & u \end{bmatrix}, \qquad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution is

$$\xi_1 = \frac{1}{3} - \frac{1}{3}(d + c + u)\xi_2, \qquad \xi_2 = \frac{u - (d + c + u)/3}{d^2 + c^2 + u^2 - (d + c + u)^2/3}.$$

We set the price of the derivative to be equal to the price of the approximately replicating portfolio:

$$\pi(H) = \pi(V) = \xi_1 + \xi_2 \pi(S).$$

If $\pi(S) = ES = (d + c + u)/3$, then $\pi(H) = EH = 1/3$.

3.2 Statistical Arbitrage and the Law of Approximate Price

Statistical arbitrage is a financial operation where a profit is obtained with a high probability. The principle of excluding the possibilities of statistical arbitrage is a pricing principle which can be used when the principle of excluding arbitrage does not apply. However, the concept of statistical arbitrage is more vague than the concept of arbitrage. Let us compare the principle of excluding arbitrage to the concept of excluding statistical arbitrage.

1. *Excluding arbitrage.* The value of a derivative is D_T at time T. Let us have an other asset whose value is A_T at time T. Assume that the values are equal with probability one: $P(D_T = A_T) = 1$. Then it should hold that the value of the derivative and the other asset are equal at all previous times: $D_t = A_t$ for all previous times t. Otherwise, there would be an arbitrage opportunity: sell the more expensive instrument and buy the cheaper instrument to obtain a risk free profit at time T.

2. *Excluding statistical arbitrage.* The value of a derivative is D_T at time T. Let us have an other asset whose value is A_T at time T. If the random variables D_T and A_T are "close", then the prices D_t and A_t should be close at all previous times t. The closeness of random variables can be defined in many ways. For example, we can say that two random variables D_T and A_T are close when $E(D_T - A_T)^2$ is small. A derivative can be priced with statistical arbitrage if we can construct an asset which replicates the payoff of the derivative with high probability.

4 Futures on a Stock

We consider a futures contract on a stock. The futures contract is made at time t and the contract specifies that the buyer of the contract has to buy the stock at a later time T with price K. We assume that the stock does not pay dividends during the time period from t to T. Let us denote with

S_t and S_T the prices of the stock at times t and T. The value of the futures contract at time T is

$$F_T = S_T - K,$$

because the buyer of the futures contract gives away K and receives S_T. We want to determine the fair value F_t at time t of the futures contract. We may replicate the futures contract buy buying the stock S_t and borrowing the amount $e^{-r(T-t)}K$, where $r > 0$ is the interest rate for the period from t to T. At time t the value of this portfolio is

$$S_t - e^{-r(T-t)}K.$$

One can see immediately that at time T the value of this portfolio is $F_T = S_T - K$ with probability one. Thus,

$$F_t = S_t - e^{-r(T-t)}K,$$

by the law of one price, to exclude arbitrage.

However, the futures contract is such that nothing changes hands at time t, and the fair forward price is called such value of K which makes the value F_t of the futures contract zero. Choosing $F_t = 0$ gives

$$K = K_t = e^{r(T-t)}S_t. \tag{2}$$

When an investor enters a futures contract in a futures exchange, this does not imply any cash flows, but the exchange requires from the investor a liquid collateral in order to secure a possible future payment. The future prices which are quoted in a futures exchange are the forward prices K_t (which are determined by the supply and demand). Numbers K_t are called futures prices or forward prices.

5 Put-Call Parity

The price of a put can always be expressed in terms of the price of a call, and conversely. We have the put-call parity:

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \tag{3}$$

where K is the common strike price of the call and put, and r is the yearly interest rate for the period from t to T. It is clear that at the expiration we have $C_T - P_T = S_T - K$. The put-call parity extends this result for times t before the expiration time T. We do not need to know fair values for C_t and P_t in order to have a formula for their difference.

5.1 Derivation of the Put-Call Parity

Consider the portfolio $V_t^{(1)}$ obtained by buying the call and writing the put:

$$V_t^{(1)} = C_t - P_t.$$

At the expiration we have $V_T^{(1)} = S_T - K$. This can be seen immediately:[4]

$$C_T - P_T = \max\{0, S_T - K\} - \max\{0, K - S_T\} = S_T - K.$$

Consider second portfolio $V_t^{(2)}$ obtained by buying the stock and borrowing the amount $Ke^{-r(T-t)}$:

$$V_t^{(2)} = S_t - Ke^{-r(T-t)}.$$

At the expiration we have $V_T^{(2)} = S_T - K$. Since with probability 1, $V_T^{(1)} = V_T^{(2)}$, we have

$$V_t^{(1)} = V_t^{(2)}$$

for all times t before T, to exclude arbitrage. This is equivalent to (3).

5.2 Consequences of the Put-Call Parity

5.2.1 Bounds for the Option Price

We have that

$$\max\left\{S_t - e^{-r(T-t)}K, 0\right\} \leq C_t \leq S_t. \tag{4}$$

Indeed, $C_t \geq 0$ is obvious, since the right to buy a stock involves no obligations. Also, $C_t \leq S_t$ is obvious, since the right to buy a stock must be less valuable than than the stock itself. The put call parity and the fact that $P_t \geq 0$ gives

$$S_t - Ke^{-r(T-t)} = C_t - P_t \leq C_t.$$

5.2.2 American Options

The put-call parity was derived for European options. However, this parity can be used to show that for a stock which does not pay dividends,

$$C_t^A = C_t^E, \tag{5}$$

[4] If $S_T \leq K$, then the call option expires worthless ($C_T = 0$), and the value of the put option is $P_T = K - S_T$. Thus in this case $C_T - P_T = S_T - K$. If $S_T \geq K$, then the value of the call option is $C_T = S_T - K$ and the put option is worthless ($P_T = 0$). Thus also in this case $C_T - P_T = S_T - K$.

where C_t^A is the price of an American call option and C_t^E is the price of an European call option. We know that $C_t^A \geq C_t^E$, because an American option has more rights than the corresponding European option. The lower bound in (4) implies

$$S_t - K \leq S_t - Ke^{-r(T-t)} \leq C_t^E \leq C_t^A,$$

where $S_t - K$ is the cash flow generated by the exercise of the call option. Since $C_t^A \geq S_t - K$, an early exercise is always suboptimal and one should sell the American call option and not exercise it. Since it is not optimal to exercise the option, the possibility for the early exercise is not worth of anything, and the American call option has the same value as the European call option.

However, an American put option is in general worth more than the corresponding European put option. For the American options we have

$$C_t^A - P_t^A < S_t - Ke^{-r(T-t)}.$$

The difference between the put and call options comes from the fact that the value of a put option increases as the price of the stock decreases. When the value of the stock decreases, then the absolute price changes become smaller. We can reach the point where the stock has so small values that further decreases in the stock price would not give a better rate of return to the put option than the risk free rate. At that point it is better to exercise the option and invest in the risk free rate. With calls the situation reverses because as the stock price increases, absolute price changes increase.

6 Black-Scholes Price

6.1 Call and Put Prices

The Black-Scholes price of the call option at time t, with strike price K, and with the maturity date T, is equal to

$$C_t(S_t, K, T) = S_t\Phi(z_+) - Ke^{-r(T-t)}\Phi(z_-), \tag{6}$$

where S_t is the stock price when the option is written, $r > 0$ is the annualized risk free rate,

$$z_\pm = \frac{\log_e(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and Φ is the distribution function of the standard Gaussian. The time to expiration $T - t$ is expressed as fractions of year. The put price is

$$P_t(S_t, K, T) = -S_t\Phi(-z_+) + Ke^{-r(T-t)}\Phi(-z_-). \tag{7}$$

Note that it can be convenient to write

$$z_\pm = \frac{\log_e\left(S_t e^{r(T-t)}/K\right) \pm (T-t)\,\sigma^2/2}{\sigma\sqrt{T-t}}.$$

The Black-Scholes prices are derived under the assumption of a log-normal distribution of the stock price: It is assumed that at time $t < T$

$$S_T \sim S_t \exp\left\{\mu(T-t) + \sigma\sqrt{T-t}\,Z_t\right\},$$

where $Z_t \sim N(0,1)$, $\mu \in \mathbf{R}$ is the drift, and $\sigma > 0$ is the volatility. The volatility σ is the only unknown parameter that need to be estimated, since μ does not appear in the price formula.

We can study the qualitative behaviour of the Black-Scholes prices as a function of five parameters σ, $T-t$, r, S_t, and K. The price of calls and puts increases as σ increases. We have

$$\lim_{\sigma\to\infty} C_t(S_t, K, T) = S_t$$

and

$$\lim_{\sigma\to 0} C_t(S_t, K, T) = \left(S_t - e^{-r(T-t)}K\right)_+,$$

which are the bounds derived form the put-call parity in (4). The prices of calls and puts increase as the time to maturity $T-t$ increases. The price of a call increases as S_t increases and the price of a call decreases as K increases, but for puts the relations reverse. The price of a call increases as the interest rate r increases but the price of a put decreases as the interest rate r increases.

6.2 Calculation of the Black-Scholes Prices

For the application of the Black-Scholes formula the time $T-t$ is taken as the time in fractions of year. For example, when the time to expiration is 20 days, then $T-t = 20/365$. Also, the risk free rate r is expressed as an annual rate.

The only unknown parameter σ has to be estimated. Let $S_{t_0}, \ldots, S_{t_n}$ be an equally spaced sample of stock prices and let us denote $\Delta t = t_i - t_{i-1}$, for $i = 1, \ldots, n$. We assume that $Y_i = \log\left(S_{t_i}/S_{t_{i-1}}\right)$, $i = 1, \ldots, n$, are i.i.d. $N(m, s^2)$. We can estimate s^2 with the sample variance

$$\hat{s}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2,$$

where $\bar{Y} = n^{-1}\sum_{i=1}^{n} Y_i$. Then an estimator of $\sigma = s(\Delta t)^{-1/2}$ is

$$\hat{\sigma} = \hat{s}\,(\Delta t)^{-1/2}. \tag{8}$$

For example, if we sample stock prices daily and use the convention that time is expressed as fractions of year, then $\Delta t = 1/250$ and $\hat{\sigma} = \hat{s}\sqrt{250}$.[5] If we sample stock prices monthly, then $\Delta t = 1/12$ and $\hat{\sigma} = \hat{s}\sqrt{12}$. The normalized sample standard deviation in (8) is called the annualized sample standard deviation.

6.3 Call and Put Prices Can Be Written as Expectations

We can write the prices of a call and a put as the expectations

$$C_t(S_t, K, T) = e^{-r(T-t)} E\left[(S_T - K)_+\right] \tag{9}$$

and

$$P_t(S_t, K, T) = e^{-r(T-t)} E\left[(K - S_T)_+\right],$$

where the expectation is taken with respect the distribution of S_T, defined by,

$$S_T = S_t \exp\left\{\mu(T-t) + \sigma\sqrt{T-t}\, Z\right\}, \tag{10}$$

where $Z \sim N(0,1)$ and

$$\mu = r - \frac{1}{2}\sigma^2.$$

Indeed, let us denote the density of the standard Gaussian distribution by $\phi(z) = (2\pi)^{-1/2} e^{-z^2/2}$, $z \in \mathbf{R}$. Then,

$$E_t(S_T - K)_+ = \int_w^\infty \left(S_t \exp\{z\sigma\sqrt{T-t} + \mu(T-t)\} - K\right)\phi(z)\, dz,$$

where

$$w = \frac{\log_e(K/S_t) - \mu(T-t)}{\sigma\sqrt{T-t}}.$$

We have

$$\exp\left\{z\sigma\sqrt{T-t}\right\}\,\phi(z) = \exp\left\{\frac{1}{2}\,\sigma^2(T-t)\right\}\,\phi\left(z - \sigma\sqrt{T-t}\right).$$

[5] The actual number of trading days in a year is between 250 and 252. There are 365 days in a year, but if we ignore the days, when there are no trading, then $\Delta t = 1/250$.

Thus,

$$\begin{aligned}
E_t(S_T-K)_+ &= S_t e^{\mu(T-t)} \int_w^\infty e^{z\sigma\sqrt{T-t}}\,\phi(z)\,dz - K\int_w^\infty \phi(z)\,dz \\
&= S_t e^{\mu(T-t)+\sigma^2(T-t)/2} \int_{w-\sigma\sqrt{T-t}}^\infty \phi(z)\,dz - K\int_w^\infty \phi(z)\,dz \\
&= S_t e^{\mu(T-t)+\sigma^2(T-t)/2}\Phi(\sigma\sqrt{T-t}-w) - K\Phi(-w). \qquad (11)
\end{aligned}$$

Similarly,

$$\begin{aligned}
E_t(K-S_T)_+ &= \int_{-\infty}^w \left(K - S_t\exp\{z\sigma\sqrt{T-t}+\mu(T-t)\}\right)\phi(z)\,dz \\
&= K\Phi(w) - S_t e^{\mu(T-t)+\sigma^2(T-t)/2}\Phi(w-\sigma\sqrt{T-t}). \qquad (12)
\end{aligned}$$

This leads to the call price

$$C_t(S_t,K,T) = e^{-r(T-t)}\left[S_t e^{r(T-t)}\Phi(z_+) - K\Phi(z_-)\right],$$

and to the put price

$$P_t(S_t,K,T) = e^{-r(T-t)}\left[-S_t e^{r(T-t)}\Phi(-z_+) + K\Phi(-z_-)\right],$$

where

$$z_\pm = \frac{\log_e\left(S_t e^{r(T-t)}/K\right) \pm (T-t)\,\sigma^2/2}{\sigma\sqrt{T-t}},$$

which are equal to the prices (6) and (7).

6.4 Derivation of the Price Using the Put-Call parity

We can derive the Black Scholes price for the calls and puts using the put-call parity given in Section 5. We assume that the distribution of the stock price S_T is defined by (10). Let us denote by C_t the value of the call option at time t. We assume that the value of the call option is equal to

$$C_t = e^{-r(T-t)}E_tC_T = e^{-r(T-t)}E_t(S_T-K)_+$$

and the value of the put option is equal to

$$P_t = e^{-r(T-t)}E_tP_T = e^{-r(T-t)}E_t(K-S_T)_+.$$

Thus, using (11) and (12),

$$C_t - P_t = S_t e^{(\mu+\sigma^2/2-r)(T-t)} - e^{-r(T-t)}K,$$

because $\Phi(x)+\Phi(-x)=1$ for all $x\in\mathbf{R}$. The put-call parity (3) implies that we have to take

$$\mu = r - \frac{1}{2}\sigma^2. \qquad (13)$$

Inserting (13) to (11) and (12) leads to (6) and (7).