Lecture Notes 3

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1 The One Step Binary Model

Let the market consist of stock S_t, bond B_t, and derivative H_t. We consider the single period model where $t = 0$ or $t = 1$.

The bond takes value 1 at $t = 0$ and value $1 + r\Delta t$ at $t = 1$:

$$B_0 = 1, \qquad B_1 = 1 + r\Delta t,$$

where $r > 0$ is the risk free rate and Δt is the length of the time period from $t = 0$ to $t = 1$.

At time $t = 0$, the value of the stock is $S_0 = s_0$ and the value S_1 is a random variable. The stock may take only two values at time 1. At $t = 1$, the value is $s_{1,0}$ with probability $1 - p$ and the value is $s_{1,1}$ with probability p:

$$S_0 = s_0, \qquad P(S_1 = s_{1,0}) = 1 - p, \qquad P(S_1 = s_{1,1}) = p.$$

We assume that

$$s_{1,0} \le s_0(1 + r\Delta t) \le s_{1.1}. \tag{1}$$

Stock S_t is the underlying of derivative H_t. The derivative takes two possible values $H_1(s_{1,0})$ and $H_1(s_{1,1})$ and we have

$$P(H_1(S_1) = H_1(s_{1,0})) = 1 - p, \qquad P(H_1(S_1) = H_1(s_{1,1})) = p.$$

We want to find a fair value $H_0 \in \mathbf{R}$ for the the derivative at $t = 0$.

2 Pricing in the One Step Binary Model

Let us replicate the derivative H_t with a portfolio

$$V_t = \xi^0 B_t + \xi^1 S_t.$$

The portfolio has ξ^0 bonds and ξ^1 stocks. At time $t = 1$ the value of the portfolio is

$$V_1 = \xi^0(1 + r\Delta t) + \xi^1 S_1.$$

To obtain $P_0(V_1 = H_1(S_1))$ we need to choose ξ^1 and ξ^2 so that

$$\begin{aligned} \xi^0(1 + r\Delta t) + \xi^1 s_{1,0} &= H_1(s_{1,0}), \\ \xi^0(1 + r\Delta t) + \xi^1 s_{1,1} &= H_1(s_{1,1}). \end{aligned}$$

The first equation leads to

$$\xi^0 = (1 + r\Delta t)^{-1} \left(H_1(s_{1,0}) - \xi^1 s_{1,0}\right). \tag{2}$$

Insertig this value of ξ^0 to the second equation gives

$$\xi^1 = \frac{H_1(s_{1,1}) - H_1(s_{1,0})}{s_{1,1} - s_{1,0}}. \tag{3}$$

The law of one price implies that the price of the option is equal to the value of the replicating portfolio at time $t = 0$:

$$H_0 = V_0 = \xi^0 + \xi^1 s_0,$$

where ξ^0 and ξ^1 are defined in (2) and (3). The price H_0 can be written as an expectation with respect to the risk neutral measure. We have

$$\begin{aligned} H_0 & \\ &= (1 + r\Delta t)^{-1} \left[H_1(s_{1,0}) - \xi^1 s_{1,0} + \xi^1(1 + r\Delta t)s_0\right] \\ &= (1 + r\Delta t)^{-1} \left[\frac{s_{1,1} - s_0(1 + r\Delta t)}{s_{1,1} - s_{1,0}} H_1(s_{1,0}) + \frac{s_0(1 + r\Delta t) - s_{1,0}}{s_{1,1} - s_{1,0}} H_1(s_{1,1})\right] \\ &= (1 + r\Delta t)^{-1} \left[(1 - q)H_1(s_{1,0}) + qH_1(s_{1,1})\right] \\ &= (1 + r\Delta t)^{-1} E_0^{(q)} H_1(S_1), \end{aligned} \tag{4}$$

where

$$q = \frac{s_0(1 + r\Delta t) - s_{1,0}}{s_{1,1} - s_{1,0}}. \tag{5}$$

Condition (1) quarantees that $0 \le q \le 1$. The probability measure which gives the probability q to the event $S_1 = s_{1,1}$ and the probability $1 - q$ to the event $S_1 = S_{1,0}$ is called a risk neutral measure because $E_0^{(q)} S_1 = s_0(1 + r\Delta t)$.

3 The Multistep Binary Model

The multistep binary model for the stock price is a discrete time model for the price process. We divide the time interval $[t, T]$ to n subintervals and denote the $n+1$ time points by

$$t_{k,n} = t + \frac{k(T-t)}{n}, \qquad k = 0, \ldots, n.$$

At time $t_{0,n} = t$ the stock takes value s_0. When the value of the stock at time $t_{k-1,n}$ is s, then at time $t_{k,n}$ the stock can take values ds and us, where $0 < d < 1 < u$. Thus, at time $t_{k,n}$, $k = 0, \ldots n$, the stock can take $k+1$ values

$$s_{k,j} = u^j d^{k-j} s_0, \qquad j = 0, \ldots, k. \tag{6}$$

The probabilities for the up and down movements are p and $1-p$:

$$P\left(S_{t_{k,n}} = su \,\middle|\, S_{t_{k-1,n}} = s\right) = p, \qquad P\left(S_{t_{k,n}} = sd \,\middle|\, S_{t_{k-1,n}} = s\right) = 1-p.$$

We choose

$$u = 1 + \sigma\sqrt{\Delta t}, \qquad d = 2 - u = 1 - \sigma\sqrt{\Delta t},$$

where $\sigma > 0$ and

$$\Delta t = \frac{T-t}{n}.$$

We choose also

$$p = \frac{1}{2} + \frac{\mu_0\sqrt{\Delta t}}{2\sigma}, \qquad 1 - p = \frac{1}{2} - \frac{\mu_0\sqrt{\Delta t}}{2\sigma}$$

where

$$\mu_0 = \mu + \frac{1}{2}\sigma^2.$$

The stock price can be written as

$$S_{t_{k,n}} = s_0 \prod_{i=1}^{k} \left(1 + w_i \sigma \sqrt{\Delta t}\right), \tag{7}$$

where $w_1, w_2, \ldots$ are such i.i.d. random variables that $w_i = 1$ when $S_{t_{i,n}}$ is a result of an up-movement and $w_i = -1$ when $S_{t_{i,n}}$ is a result of a down-movement, so that

$$P(w_i = 1) = p, \qquad P(w_i = -1) = 1 - p.$$

4 Pricing in the Multistep Binary Model

The evolution of the stock price has been described using a recombining binary tree in a multistep binary model. The price in a multistep binary model can be found by backward induction. We know the price at the expiration time $T = t_{n,n}$. We can use the single period model to calculate the price at time $t_{n-1,n}$ and go backwards step by step to obtain the price at time $t = t_{0,n}$. The price $H_t(s_0)$ of the derivative is calculated using backwards induction with the following steps.

1. At time $T = t_{n,n}$ the prices of the derivative are given by $H_T(s_{n,j})$, $j = 0, \ldots, n$.

2. At time $t_{k-1,n}$, $k = 1, \ldots, n$, when we are at price $s_{k-1,j}$, then we know that the two possible prices for the derivative are $H_{t_{k,n}}(s_{k,j})$ and $H_{t_{k,n}}(s_{k,j+1})$. We can use the single period model to calculate the price at time $t_{k-1,n}$. We get the price from (4) as

$$\begin{aligned} & H_{t_{k-1,n}}(s_{k-1,j}) \\ & = (1 + r\Delta t)^{-1} \left[(1-q) H_{t_{k,n}}(s_{k,j}) + q H_{t_{k,n}}(s_{k,j+1}) \right], \end{aligned} \tag{8}$$

where

$$q = \frac{s_{k-1,j}(1 + r\Delta t) - s_{k,j}}{s_{k,j+1} - s_{k,j}} = \frac{1}{2} + \frac{r\sqrt{\Delta t}}{2\sigma},$$

where the prices are

$$s_{k,j+1} = s_{k-1,j}(1 + \sigma\sqrt{\Delta t}), \qquad s_{k,j} = s_{k-1,j}(1 - \sigma\sqrt{\Delta t}),$$

and $\Delta t = (T - t)/n$.

By induction we obtain the following theorem.

Theorem 1 *The price of the derivative at time t is*

$$H_t(s_0) = (1 + r\Delta t)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} H_T(s_{n,j}), \tag{9}$$

where the possible prices of the stock at the expiration are given in (6) as

$$s_{n,j} = u^j (2-u)^{n-j} s_0, \qquad j = 0, \ldots, n.$$

with $u = 1 + \sigma\sqrt{\Delta t}$.

Proof of Theorem 1. Let us make the induction hypothesis that the claim holds for binary trees with $n-1$ levels. Then, we can price the derivative at time $t_{1,n}$:

$$
\begin{aligned}
H_{t_{1,n}}(s_{1,0}) &= (1+r\Delta t)^{-n+1} \sum_{j=0}^{n-1} \binom{n-1}{j} q^j (1-q)^{n-j-1} H_T(s_{n,j}), \\
H_{t_{1,n}}(s_{1,1}) &= (1+r\Delta t)^{-n+1} \sum_{j=0}^{n-1} \binom{n-1}{j} q^j (1-q)^{n-j-1} H_T(s_{n,j+1}).
\end{aligned}
$$

Now we get, applying the one step binomial model, that at time $t = t_{0,n}$

$$
\begin{aligned}
H_t(s_0) &= (1+r\Delta t)^{-1} \left[(1-q)H_{t_{1,n}}(s_{1,0}) + qH_{t_{1,n}}(s_{1,1})\right] \\
&= (1+r\Delta t)^{-n} \left[\binom{n-1}{0} q^0 (1-q)^n H_n(s_{n,0}) \right. \\
&\quad + \sum_{j=1}^{n-1} \left[\binom{n-1}{j} + \binom{n-1}{j-1}\right] q^j (1-q)^{n-j} H_n(s_{n,j}) \\
&\quad \left. + \binom{n-1}{n-1} q^n (1-q)^0 H_n(s_{n,n}) \right] \\
&= (1+r\Delta t)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} H_n(s_{n,j}),
\end{aligned}
$$

where we used the fact

$$
\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}.
$$

We have proved Theorem 1.

The price in (9) can be written as the expectation

$$
H_t(s_0) = (1+r\Delta t)^{-n} E_t H_T(S_T), \qquad (10)
$$

where S_T is a random variable taking values $s_{n,j}$ and

$$
P(S_T = s_{n,j}) = \binom{n}{j} q^j (1-q)^j, \qquad j = 0, \ldots, n.
$$

5 Asymptotics of the Multistep Binary Model

5.1 Asymptotic Normality in the Multistep Binary Model

In the n-step binary model the stock price S_T was written at step n in (7) as

$$
S_T = s_0 \prod_{i=1}^{n} \left(1 + w_i \sigma \sqrt{\Delta t}\right),
$$

where $\Delta t = (T-t)/n$, w_i are i.i.d. random variables with

$$P(w_i = 1) = \frac{1}{2} + \frac{\mu_0\sqrt{\Delta t}}{2\sigma}, \qquad P(w_i = -1) = \frac{1}{2} - \frac{\mu_0\sqrt{\Delta t}}{2\sigma},$$

and

$$\mu_0 = \mu + \frac{1}{2}\sigma^2.$$

We can show that

$$\log_e\left(\frac{S_T}{s_0}\right) \xrightarrow{d} N\left(\mu(T-t), \sigma^2(T-t)\right), \tag{11}$$

as $n \to \infty$.

Let us denote $X_i = w_i\sigma\sqrt{\Delta t}$. We can write

$$\log_e\left(\frac{S_T}{s_0}\right) = \sum_{i=1}^{n} \log_e(1+X_i) = Q_n - \frac{1}{2}R_n + S_n,$$

where $Q_n = \sum_{i=1}^n X_i$, $R_n = \sum_{i=1}^n X_i^2$, and $S_n = \sum_{i=1}^n X_i^2 r(X_i)$, where we denote $r(x) = x^{-2}[\log(1+x) - x - x^2/2]$. Now it holds that $\lim_{x\to 0} r(x) = 0$. We have that

1. $Q_n \xrightarrow{d} N(\mu_0(T-t), \sigma^2(T-t))$,
2. $R_n \xrightarrow{p} \sigma^2(T-t)$,
3. $S_n \xrightarrow{p} 0$,

as $n \to \infty$. Because $\mu_0 = \mu + \sigma^2/2$, the claim (11) follows from items 1–3.

To prove item 1, we note that

$$Ew_i = \frac{\mu_0\sqrt{\Delta t}}{\sigma}, \qquad Ew_i^2 = 1, \qquad \mathrm{Var}(w_i) = 1 - \left(\frac{\mu_0\sqrt{\Delta t}}{\sigma}\right)^2.$$

Thus,

$$EX_i = \mu_0\Delta t, \qquad \mathrm{Var}(X_i) = \sigma^2\Delta t\left[1 - \left(\frac{\mu_0\Delta t}{\sigma}\right)^2\right].$$

We have that

$$nEX_i = \mu_0(T-t), \qquad n\mathrm{Var}(X_i) \to \sigma^2(T-t),$$

as $n \to \infty$. Thus item 1 follows by the central limit theorem.

To prove item 2, we use the fact that $EX_i^2 = \sigma^2 \Delta t$ which implies that $nEX_i^2 = \sigma^2(T-t)$. Thus the weak law of the large numbers implies item 2.

To prove item 3, we note that

$$|S_n| \le R_n \max_{i=1,\ldots,n} |r(X_i)| \le R_n \max\{r(\sigma\Delta t), r(-\sigma\Delta t)\}.$$

Item 2 imples that $R_n = O_p(1)$ and since $\lim_{x\to 0} r(x) = 0$, we have that $\max\{r(\sigma\Delta t), r(-\sigma\Delta t)\} = o(1)$, so that $S_n = o_p(1)$.

5.2 Convergence to the Black-Scholes Price

The price $H_t(s_0)$, defined in (9), is called the Cox-Ross-Rubinstein price. We can prove that the Cox-Ross-Rubinstein price converges to the Black-Scholes price as $n \to \infty$. This follows from the general theorem, which states that if $X_n \xrightarrow{d} X$ and $f : \mathbf{R} \to \mathbf{R}$ is bounded and continuous, then $Ef(X_n) \to Ef(X)$, as $n \to \infty$.

First, we have noted in (10) that

$$H_t(s_0) = (1 + r\Delta t)^{-n} E_t H_T(S_T).$$

Second, we have proved the convergence in (11), which states that

$$S_T \xrightarrow{d} s_0 \exp\left\{(r - \sigma^2/2)(T-t) + \sigma\sqrt{T-t}\, Z\right\},$$

where $Z \sim N(0,1)$, when $\mu_0 = r$, so that $\mu = r - \sigma^2/2$. Combining the results gives

$$\lim_{n\to\infty} H_t(s_0) = e^{-r(T-t)} E H_T(X), \tag{12}$$

where

$$\log_e\left(\frac{X}{s_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right), \tag{13}$$

and we used the fact that $(1 + r\Delta t)^{-n} \to e^{-r(T-t)}$, because $\Delta t = (T-t)/n$.

Choosing

$$H_T(x) = (K - x)_+,$$

shows that the Cox-Ross-Rubinstein price of a European put option converges to the Black-Scholes price of a European put option, because the function $x \mapsto (K-x)_+$, $x \ge 0$, is bounded and continuous, and the Black-Scholes price is written in Lecture Notes 2 as an expectation with respect to distribution (13). The convergence for the call options cannot be inferred directly, because the function $x \mapsto (x-K)_+$, $x \ge 0$, is not bounded. However, we can use the put-call parity to conclude that the Cox-Ross-Rubinstein prices have to satisfy

$$C_t(s_0) - P_t(s_0) = S_t - K(1 - r\Delta t)^{-n}.$$

Since

$$\lim_{n\to\infty} P_t(s_0) = e^{-r(T-t)} E_t P_T(X),$$

it holds that

$$\lim_{n\to\infty} C_t(s_0) = e^{-r(T-t)} E_t P_T(X) + S_t - K e^{-r(T-t)} = E_t C_T(X).$$

6 American Put Options

The American call options have the same price as the European call options, as shown in Lecture Notes 2. Thus, the American call options can be priced similarily as the European call options using the Black-Scholes prices or the recombining binary trees. However, the American put options have to priced differently, by taking into account the possibility of an early exercise. We can use the recombining binary tree to price the American put options. At every node of the tree we consider whether it is better to exercise or to keep the option for a future exercise. We are not able to obtain a closed form formula for the price of the American options, but we obtain an algorithm for the calculation of the price. First, single step binary model is studied. Second, multistep binary model leads to the final algorithm.

6.1 American Put Options in the One Step Binary Model

In the one step binary model the American put option can be exercised at time $t = 0$ or at time $t = 1$. Let us denote with P_0^A the value of the American put option at time $t = 0$ and let us denote with P_0^E the value of the European put option at time $t = 0$. An arbitrage argument shows that

$$P_0^A = \max\left\{K - s_0, P_0^E\right\}.$$

The value of the European put option was calculated previously and we obtained

$$P_0^E = (1 + r\Delta t)^{-1} E_q (K - S_1)_+$$

where

$$E_q(K - S_1)_+ = (1 - q)(K - s_{1,0})_+ + q(K - s_{1,1})_+$$

with

$$q = \frac{s_0(1 + r\Delta t) - s_{1,0}}{s_{1,1} - s_{1,0}}.$$

6.2 American Put Options in the Multistep Binary Model

The price of an American put option is determined in the n-step binomial model by recursion. Remember that in the n-step binomial model the possible prices at step k, $k = 0, \dots, n$, are

$$s_{k,j} = u^j (2-u)^{k-j} s_0, \qquad j = 0, \dots, k$$

with

$$u = 1 + \sigma\sqrt{\Delta t}.$$

The recursive steps are the following.

1. At time $T = t_{n,n}$ the prices of the American put option are given by

   $$H_T(s_{n,j}) = \max\left\{K - s_{n,j}, 0\right\},$$

   $j = 0, \dots, n$.

2. At time $t_{k-1,n}$, $k = 1, \dots, n$, when the stock has price $s_{k-1,j}$, we know from the previous steps of the algorithm that the two possible prices for the derivative at time $t_{k,n}$ are $H_{t_{k,n}}(s_{k,j})$ and $H_{t_{k,n}}(s_{k,j+1})$. We can use the single period model to calculate the price at time $t_{k-1,n}$. We get the price from the one-step binary model as

   $$H_{t_{k-1,n}}(s_{k-1,j}) = \max\left\{K - s_{k-1,j}, E_q H_{t_{k,n}}\left(S_{t_{k,n}}\right)\right\},$$

   where

   $$E_q H_{t_{k,n}}(S_{t_{k,n}}) = (1 + r\Delta t)^{-1}\left[(1-q)H_{t_{k,n}}(s_{k,j}) + qH_{t_{k,n}}(s_{k,j+1})\right]$$

   and

   $$q = \frac{s_{k-1,j}(1 + r\Delta t) - s_{k,j}}{s_{k,j+1} - s_{k,j}} = \frac{1}{2} + \frac{r\sqrt{\Delta t}}{2\sigma},$$

   with $\Delta t = (T-t)/n$.