Lecture Notes 4

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1 Hedging

The writer of an option hedges his position by trading with the underlying stock and the bank account. Let the option be written at time 0 and let the expiration time be T. The position can be hedged at times $t = 0, 1, \ldots, T-1$. The wealth process of the writer of the option is

$$W_t = \xi^0_{t+1} B_t + \xi_{t+1} S_t,$$

where the ξ_{t+1} is called the hedging coefficient. In the binomial model and in the Black-Scholes the perfect hedging is possible, so that we obtain $P_0(W_T = C_T) = 1$, where C_T is the value of the option at the expiration. In the other models we try to choose the hedging coefficients so that $E(W_T - C_T)^2$ is as small as possible.

1.1 Wealth and Value Processes

The price vector is denoted by

$$\bar{S}_t = (S^0_t, S_t) = (S^0_t, S^1_t, \ldots, S^N_t), \qquad t = 0, 1, \ldots, T,$$

where the components are non-negative and S^0 is the bank account. We choose $S^0_t = (1 + r)^t$, where $r > 0$ is a constant. A trading strategy is

$$\bar{\xi}_t = (\xi^0_t, \xi_t) = (\xi^0_t, \xi^1_t, \ldots, \xi^N_t), \qquad t = 1, \ldots, T.$$

The value ξ^i_t expresses the quantity of the shares of the ith asset held between $t - 1$ and t. The vector $\bar{\xi}_t$ is determined at time $t - 1$, using the information available at time $t - 1$.

1.1.1 Wealth Process

The initial investment is

$$W_0 = \bar{\xi}_1 \cdot \bar{S}_0.$$

At time $t-1$ the investment is equal to

$$W_{t-1} = \bar{\xi}_t \cdot \bar{S}_{t-1}.$$

At time t the value of the portfolio changes to

$$W_t = \bar{\xi}_t \cdot \bar{S}_t. \tag{1}$$

A trading strategy $\bar{\xi}_t$ is self-financing if

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t, \qquad t = 1, \ldots, T-1. \tag{2}$$

When the trading strategy is self-financing then only the available wealth is invested. The portfolio is rebalanced and the new portfolio vector $\bar{\xi}_{t+1}$ allocates the wealth W_t among the assets in a new way, and we get

$$W_t = \bar{\xi}_{t+1} \cdot \bar{S}_t.$$

The wealth at time t can be written as

$$\begin{aligned} W_t &= W_0 + \sum_{k=1}^{t} (W_k - W_{k-1}) \\ &= W_0 + \sum_{k=1}^{t} \left(\bar{\xi}_k \cdot \bar{S}_k - \bar{\xi}_{k-1} \cdot \bar{S}_{k-1} \right) \\ &= W_0 + \sum_{k=1}^{t} \bar{\xi}_k \cdot \left(\bar{S}_k - \bar{S}_{k-1} \right), \end{aligned} \tag{3}$$

where we used the facts $W_k = \bar{\xi}_k \cdot \bar{S}_k$ and $W_{k-1} = \bar{\xi}_{k-1} \cdot \bar{S}_{k-1} = \bar{\xi}_k \cdot \bar{S}_{k-1}$, which follows from (2).

1.1.2 Value Process

We have assumed that the rebalancing is made under the self-financing condition, so that $\bar{\xi}_{t+1}$ is chosen so that the wealth W_t is allocated among the assets. Thus, $W_t = \bar{\xi}_{t+1} \bar{S}_t$, which sets a linear constraint on the vector ξ_{t+1}. It is convenient to write the wealth process so that the ξ_t^0 is eliminated, and this can be done using the value process.

We choose the bank account asset as numéraire. The discounted price process is defined by

$$X_t^i = \frac{S_t^i}{S_t^0}, \qquad t = 0, \ldots, T, \qquad i = 0, \ldots, N.$$

We denote $X_t = (X_t^1, \ldots, X_t^N)$ and $\bar{X}_t = (X_t^0, X_t) = (1, X_t)$. The self-financing property in (2) implies that the discounted price process satisfies

$$\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t, \qquad t = 1, \ldots, T-1. \tag{4}$$

The value process is defined as

$$V_t = \frac{W_t}{S_t^0}, \qquad t = 0, \ldots, T.$$

We have that $V_0 = W_0$ because $S_0^0 = 1$. It holds that

$$V_0 = \bar{\xi}_1 \cdot \bar{X}_0, \qquad V_t = \bar{\xi}_t \cdot \bar{X}_t, \qquad t = 1, \ldots, T.$$

The value at time t can be written as

$$\begin{aligned} V_t &= V_0 + \sum_{k=1}^{t} (V_k - V_{k-1}) \\ &= V_0 + \sum_{k=1}^{t} \left(\bar{\xi}_k \cdot \bar{X}_k - \bar{\xi}_{k-1} \cdot \bar{X}_{k-1} \right) \\ &= V_0 + \sum_{k=1}^{t} \bar{\xi}_k \cdot \left(\bar{X}_k - \bar{X}_{k-1} \right), \end{aligned}$$

because $V_k = \bar{\xi}_k \cdot \bar{X}_k$ and $V_{k-1} = \bar{\xi}_{k-1} \cdot \bar{X}_{k-1} = \bar{\xi}_k \cdot \bar{X}_{k-1}$, where we used (4). We have that $X_k^0 = X_{k-1}^0 = 1$ and thus

$$\bar{\xi}_k \cdot \left(\bar{X}_k - \bar{X}_{k-1} \right) = \xi_k \cdot (X_k - X_{k-1}).$$

This implies that the value process can be written as

$$V_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}). \tag{5}$$

The wealth is obained from the value process as

$$W_t = S_t^0 V_t = S_t^0 V_0 + \sum_{k=1}^{t} S_t^0 \xi_k \cdot (X_k - X_{k-1}).$$

When $S_t^0 = (1+r)^t$, then $S_t^0 X_k = S_{t-k}^0 S_k$ and $S_t^0 X_{k-1} = S_{t-k+1}^0 (1+r) S_{k-1}$, so that

$$\begin{aligned} W_t &= S_t^0 W_0 + \sum_{k=1}^{t} S_{t-k}^0 \xi_k \cdot [S_k - (1+r)S_{k-1}] \\ &= (1+r)^t W_0 + \sum_{k=1}^{t} (1+r)^{t-k} \xi_k \cdot [S_k - (1+r)S_{k-1}]. \end{aligned}$$

1.2 Hedging in the Binomial Model

The hedging coefficient ξ_1 at time $t = 0$ is

$$\xi_1 = \frac{H_{t_{1,n}}(s_{1,1}) - H_{t_{1,n}}(s_{1,0})}{s_{1,1} - s_{1,0}},$$

where $s_{1,1} = s_0 u$, $s_{1,0} = s_0 d$, $t_{1,n} = \Delta t = T/n$, $u = 1 + \sigma\sqrt{\Delta t}$, $d = 1 - \sigma\sqrt{\Delta t}$, so that

$$s_{1,1} - s_{1,0} = 2 s_0 \sigma \sqrt{\Delta t}.$$

The writer receives the premium. The premium is invested in the bank account, the amount $\xi_1 s_0$ is borrowed from the bank account, and the amount $\xi_1 s_0$ is invested in the stock.

1.3 Hedging in the Black-Scholes Model

The delta is the derivative of the price function $C(S)$ with respect to the underlying:

$$C_s = \frac{\partial C(S)}{\partial S}.$$

In the Black-Scholes model the delta is equal to the hedging coefficient. For the Black-Scholes pricing functions the call delta is equal to

$$C_s = \Phi(z_+), \tag{6}$$

and the put delta is equal to

$$P_s = -\Phi(-z_+).$$

Thus,

$$0 < C_s < 1, \qquad -1 < P_s < 0.$$

Let us calculate the call option delta. Now the pricing function is

$$C(S) = S\Phi(z_+) - Ke^{-r(T-t)}\Phi(z_-), \tag{7}$$

where

$$z_{\pm} = \frac{\log_e(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The delta of a call is given in (6) because

$$\frac{\partial C(S)}{\partial S} = \Phi(z_+) + e^{-r(T-t)} \left[S e^{r(T-t)} \frac{\partial \Phi(z_+)}{\partial S} - K \frac{\partial \Phi(z_-)}{\partial S} \right],$$

$$\frac{\partial \Phi(z_+)}{\partial S} = \phi(z_+) \cdot \frac{\partial z_+}{\partial S} = \phi(z_+) \cdot \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{S},$$

$$\frac{\partial \Phi(z_-)}{\partial S} = \phi(z_-) \cdot \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{S},$$

and finally

$$e^{r(T-t)}\phi(z_+) = \frac{K}{S}\,\phi(z_-).$$

1.4 Wealth Distribution under Hedging

We illustrate hedging in the case of a call option with the payoff $(S_T - K)_+$.

1.4.1 Daily Hedging and the Effect of Drift

We illustrate hedging of an European call option when the stock price follows a geometric Brownian motion and delta hedging is done daily. We study the effect of drift to the hedging, and try values $\mu = 8\%$, $\mu = 50\%$, and $\mu = -8\%$ for the values of the annualized drift. The annualaized volatility is $\sigma = 15\%$ and the initial stock price is $S_0 = 100$. The strike price is $K = 105$, there are $T = 60$ days to maturity, and the risk free rate is $r = 0$. We simulated $M = 5000$ trajectories.[1]

Figure 1 shows the case where the annualized drift is $\mu = 8\%$. Panel (a) shows a histogram made from M realizations of the random variable $X_T = e^{r(T-t)}C_0 - (S_T - K)_+$, where C_0 is the Black Scholes price. The random variable X_T is the wealth of the writer of the call option at the expiration when no hedging is done. In this case the final wealth of the writer is equal to the receivings from the option premium minus the pay-off of the option. Panel (b) shows a histogram made from M realizations of the random variable $X_T + W_T$, where W_T is the final wealth obtained with the delta hedging. Thus,

[1] Trajectories of stock prices are simulated by first generating T independent $N(\mu_0, \sigma_0^2)$ random variables z_i, where $\mu_0 = \mu/250$ and $\sigma_0 = \sigma/\sqrt{250}$. Then the stock prices are generated by $S_0 = 100$ and $S_i = (1 + z_i)S_{i-1}$ for $i = 1, \ldots, T$. This is done M times to get M trajectories.

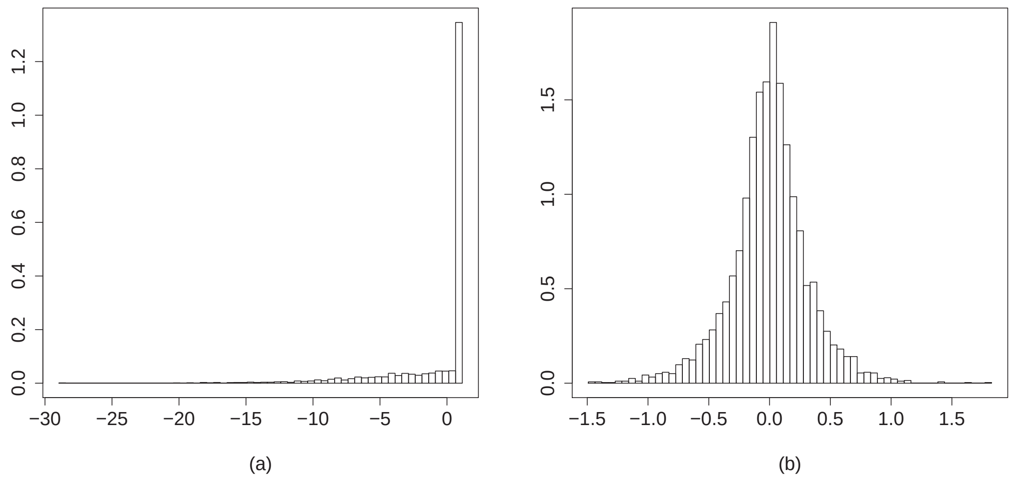

Figure 1: *Delta hedging with a moderate positive drift.* (a) A histogram from the realizations of $e^{rT}C_0 - (S_T - K)_+$, where C_0 is the option premium. (b) A histogram from the realizations of $e^{rT}C_0 - (S_T - K)_+ + W_T$, where W_T is the final value of the hedging process.

$X_T + W_T$ is the wealth of the writer of the call option at the expiration when the option is hedged. The histograms have 80 bins. Panel (a) shows that the probability distribution of wealth without hedging is highly unsymmetric: there is a large probability of a small profit and a small probability of a large loss. Panel (b) shows that the probability distribution of wealth when hedging is done is symmetric and concentrated around zero.

Figure 2 shows the setting of Figure 1 when the annualized drift is 50%, instead of 8%. Panel (a) shows the wealth of the writer at the expiration when no hedging is done and panel (b) shows the wealth when delta hedging is done daily. In this case the call option gives a profit to its owner with a large probability but this does not affect much the final wealth of the writer when hedging is done.

Figure 3 shows the setting of Figure 1 when the annualized drift is -8%, instead of 8%. Panel (a) shows the wealth of the writer at the expiration when no hedging is done and panel (b) shows the wealth when delta hedging is done daily. In this case the writer of the call option gets a profit with a large probability, but this does not affect much the final wealth of the writer when hedging is done.

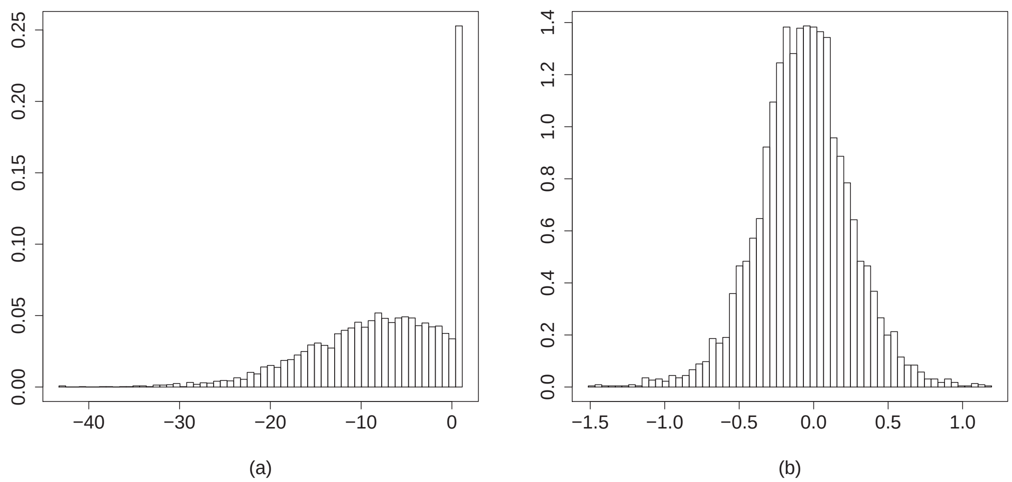

Figure 2: *Delta hedging with a large positive drift.* The probability distributions of the wealth with the setting of Figure 1 when the annualized drift is 50%, instead of 8%. (a) A histogram from the realizations of $e^{r(T-t)}C_0 - (S_T - K)_+$. (b) A histogram from the realizations of $e^{r(T-t)}C_0 - (S_T - K)_+ + W_T$.

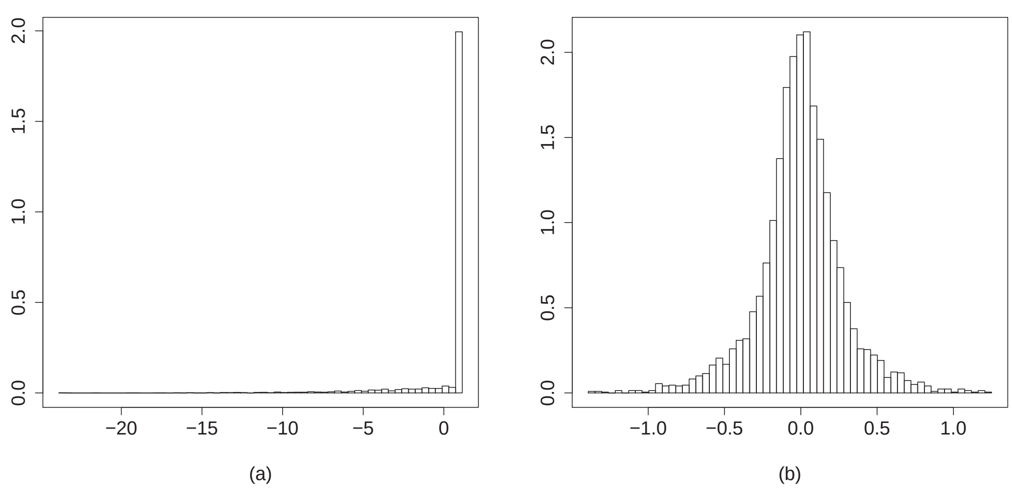

Figure 3: *Delta hedging with a negative drift.* The setting is the same as in Figure 1, but the annualized drift is −8%, instead of +8%. (a) A histogram from the realizations of $e^{r(T-t)}C_0 - (S_T - K)_+$. (b) A histogram from the realizations of $e^{r(T-t)}C_0 - (S_T - K)_+ + W_T$.

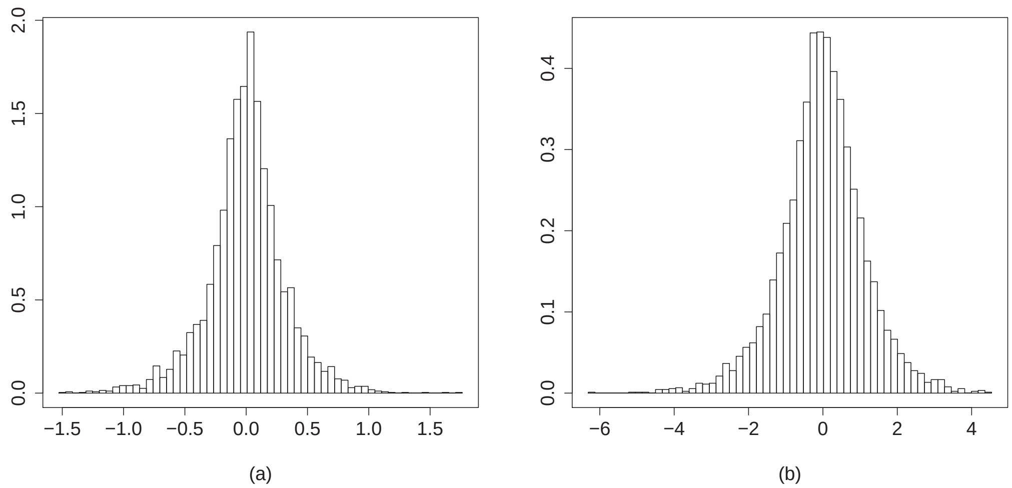

Figure 4: *Delta hedging with different volatilities.* Histograms of the final wealth of the writer with daily hedging. (a) Volatility is 15%. (b) Volatility is 50%.

1.4.2 Daily Hedging and the Effect of Volatility

We study the effect of volatility to the daily hedging of an European call option, and try values $\sigma = 15\%$, and $\sigma = 50\%$ for the values of the annualized volatility. The annualized mean is $\mu = 8\%$ and the initial stock price is $S_0 = 100$. The strike price is $K = 105$, there are $T = 60$ days to maturity, and the risk free rate is $r = 0$. We simulated $M = 5000$ trajectories from a log-normal process.

Figure 4 shows the histograms made from M realizations of the random variable $e^{r(T-t)}C_0 - (S_T - K)_+ + W_T$, where C_0 is the Black-Scholes price, and W_T is the final wealth obtained with the delta hedging. Panel (a) has the annualized volatility 15% and panel has 15%. The histograms have 80 bins. We see that the larger volatility makes the dispersion of the probability distribution of the final wealth larger.

1.4.3 Convergence of Hedging

Figure 5 illustrates the effect of hedging frequency, showing the probability distribution of the wealth of the writer at the expiration for three different hedging frequencies. The parameters are the same as in Figure 1. Panel (a) shows the case where the hedging is done once, at the time of writing the option. Panel (b) shows the case where the hedging is done twice, at time 0

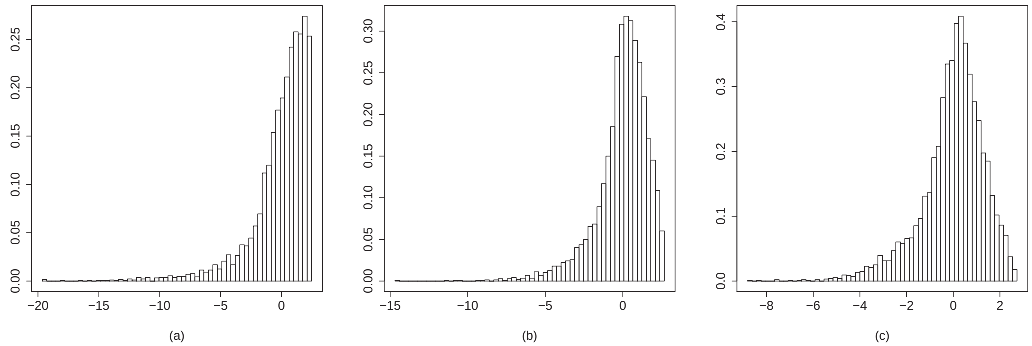

Figure 5: *Hedging with small frequencies.* (a) Probability distribution of the wealth of the writer at the expiration when hedging is done once, (b) hedging is done twice, and (c) hedging is done three times.

and at time $T/2$. Panel (c) shows the case where the hedging is done three times: at 0, $T/3$, and $2T/3$.

Figure 6 shows the convergence of the mean squared hedging error as a function of hedging frequency. Time to expiration is 64 days. Panel (a) shows the cases when the annualized mean is 8% (black line with label "1"), 50% (red line with label "2"), and −8% (green line with label "3"). The annualized volatility is 15% in all cases. Panel (b) shows the cases when the annulaized volatility is 15% (black line with label "1"), 50% (red line with label "2"), and 5% (green line with label "3"). The annualized mean is 8% in all cases.

1.4.4 Hedging of S&P 500 Calls

Figure 7 studies hedging of S&P 500 call options. Panel (a) shows a histogram made from realizations of the random variable $X_T = e^{r(T-t)}C_0 - (S_T - K)_+$, where C_0 is the Black Scholes price. Panel (b) shows a histogram made from realizations of the random variable $X_T + W_T$, where W_T is the final wealth obtained with the delta hedging. Comparing panel (b) with the histogram in Figure 1 shows that in the case of the S&P 500 data the left tail of the probability distribution of the final has heavier tail than the probability distribution with the simulated data.

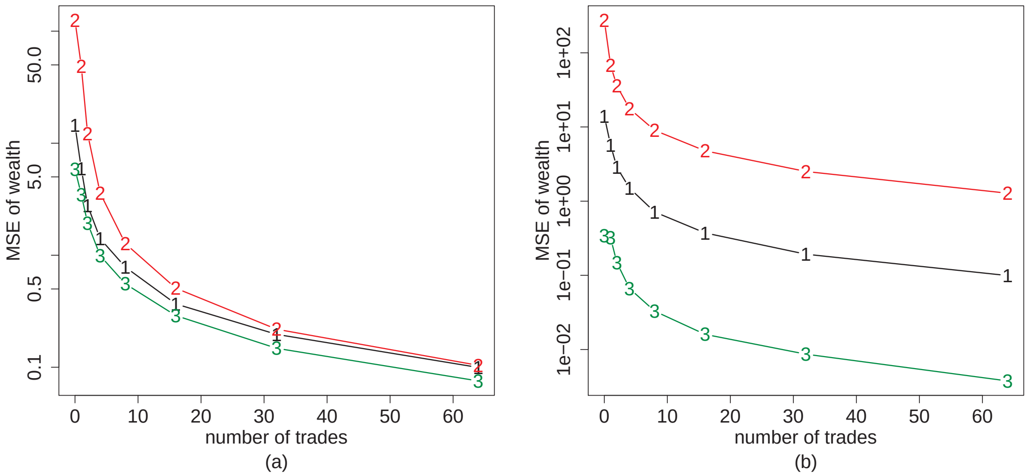

Figure 6: *Convergence of the mean squared hedging error.* (a) The annualized mean is 8% (black line with label "1"), 50% (red line with label "2"), and −8% (green line with label "3"). The annualized volatility is 15% in all cases. (b) The annulaized volatility is 15% (black line with label "1"), 50% (red line with label "2"), and 5% (green line with label "3"). The annualized mean is 8% in all cases.

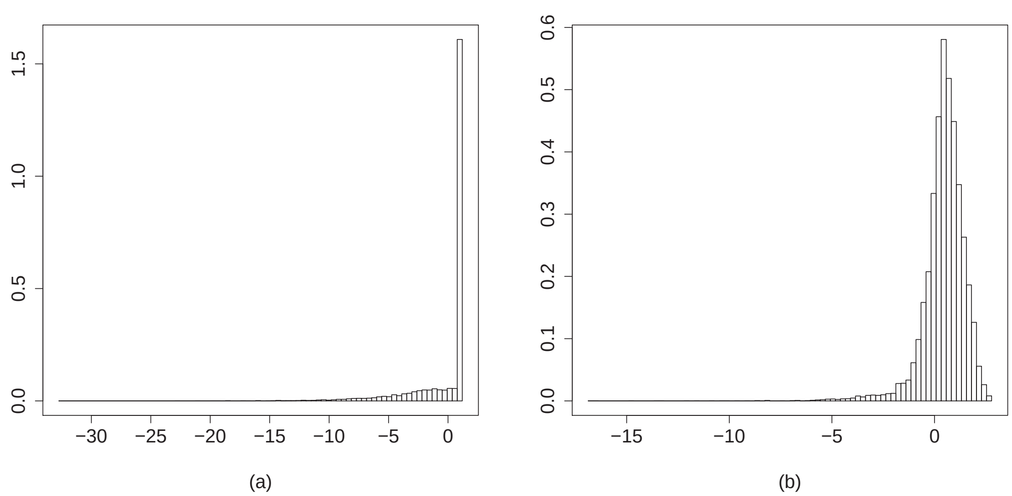

Figure 7: *Hedging with S&P 500 data.* (a) A histogram from the realizations of $e^{rT}C_0 - (S_T - K)_+$, where C_0 is the option premium. (b) A histogram from the realizations of $e^{rT}C_0 - (S_T - K)_+ + W_T$, where W_T is the final value of the hedging process.