Lecture Notes 5

Jussi Klemelä

October, 2012

1 Quadratic Hedging and Dynamic Optimization

The exact solution for the quadratic hedging can be given using dynamic optimization. We present the solution first for the one period model and the for the two period model.

1.1 One Period Model

We consider the pricing of an European option in the single period model. In the single period model the option is hedged only at the time of writing the option.

1.1.1 Definition of the One Period Model

The single period model is the following.

- The underlying security has value S_0 at the beginning of the period and value S_1 at the expiration of the option. The price S_0 is a fixed number and S_1 is a random variable.

- At time zero the option seller sells the option at price C_0. The value of the European option at the expiration is denoted by C_1. For example, in the case of a call option $C_1 = \max\{S_1 - K, 0\}$, where K is the strike price.

- The time between the beginning of the period and the end of the period is denoted by Δt and is expressed in fractions of a year. The annual risk free rate is denoted by r. The value of a bank account increases by multiplying with $1 + r\Delta t$.

1.1.2 Pricing in the One Period Model

The initial wealth is W_0. This amount is invested in the bank account. The amount ξS_0 is borrowed at the risk free rate and this money is invested in stock, so that there are ξ stocks in the portfolio. The value of the portfolio at time 1 is

$$W_1 = (1 + r\Delta t)W_0 + \xi\,(S_1 - (1 + r\Delta t)S_0).$$

We find values of W_0 and ξ that minimize

$$E_0(W_1 - C_1)^2. \tag{1}$$

This is equivalent to finding values W_0 and ξ that minimize

$$E_0(V_1 - H_1)^2,$$

where $V_1 = (1 + r\Delta t)^{-1}W_1$ and $H_1 = (1 + r\Delta t)^{-1}C_1$. Let us write

$$V_1 = W_0 + \xi\Delta X_1,$$

where

$$\Delta X_1 = (1 + r\Delta t)^{-1}S_1 - S_0. \tag{2}$$

We need to find W_0 and ξ minimizing

$$E_0(W_0 + \xi\Delta X_1 - H_1)^2.$$

Derivating with respect to W_0 and ξ we get the equations

$$\begin{cases} E_0(W_0 + \xi\Delta X_1 - H_1) = 0, \\ E_0[(W_0 + \xi\Delta X_1 - H_1)\,\Delta X_1] = 0. \end{cases}$$

The solutions are given by[1]

$$\xi = \frac{\mathrm{Cov}_0(\Delta X_1, H_1)}{\mathrm{Var}_0(\Delta X_1)}, \qquad W_0 = E_0H_1 - \xi E_0\Delta X_1. \tag{3}$$

[1]From the second equation we get

$$\xi = \frac{E_0[(H_1 - W_0)\Delta X_1]}{E_0\Delta X_1^2}$$

From the first equation we get $W_0 = E_0H_1 - \xi E_0\Delta X_1$. Inserting this gives

$$\xi\left(1 - \frac{(E_0\Delta X_1)^2}{E_0\Delta X_1^2}\right) = \frac{E_0[(H_1 - E_0H_1)\Delta X_1]}{E_0\Delta X_1^2},$$

which gives the result because $1 - (E_0\Delta X_1)^2/E_0\Delta X_1^2 = \mathrm{Var}_0(\Delta X_1)/E_0\Delta X_1^2$ and $E_0[(H_1 - E_0H_1)\Delta X_1] = \mathrm{Cov}_0(\Delta X_1, H_1)$.

The hedging coefficient and the fair price can also be written as

$$\xi = \frac{\mathrm{Cov}_0(S_1, C_1)}{\mathrm{Var}_0(S_1)} \tag{4}$$

and

$$C_0 = (1 + r\Delta t)^{-1} \left[E_0 C_1 - \xi E_0 (S_1 - (1 + r\Delta t) S_0) \right]. \tag{5}$$

Note that for the call option $\xi \in [0, 1]$ and for the put option $\xi \in [-1, 0]$ [2]

1.2 Two Period Model

We consider the case of hedging an European option at two time points (two period hedging).

1.2.1 Definition of the Two Period Model

The two period model is the following.

- The underlying security takes values S_0, S_1, and S_2 at consecutive time points $0, 1, 2$. The option is written at time 0 and it expires at time 2. The price S_0 is a fixed number and S_1 and S_2 are random variables.
- At time zero the writer of the option sells the option at price H_0. The value of the European option at the expiration is denoted by C_2. For example, in the case of a call option $C_2 = \max\{S_2 - K, 0\}$, where K is the strike price.
- The time between the time points 0, 1, and 2 is denoted by Δt and is expressed in fractions of a year. The annual risk free rate with simple combounding is denoted by r.

1.2.2 Pricing in the Two Period Model

We consider a portfolio with initial wealth W_0, which is invested in the risk free rate, and the stock is traded at time points 0 and 1. The wealth at time 1 is

$$W_1 = (1 + r\Delta t)(W_0 + \xi_1 \Delta X_1)$$

and the wealth at time 2 is

$$W_2 = (1 + r\Delta t)^2 (W_0 + \xi_1 \Delta X_1 + \xi_2 \Delta X_2),$$

[2] By the Cauchy-Schwartz inequality $|\mathrm{Cov}(S_1, H_1)| \le \sqrt{\mathrm{Var}(S_1)\mathrm{Var}(H_1)}$. For the call option $\mathrm{Var}(H_1) \le \mathrm{Var}(S_1 - K) = \mathrm{Var}(S_1)$ and for the put option $\mathrm{Var}(H_1) \le \mathrm{Var}(K - S_1) = \mathrm{Var}(S_1)$. For the call option $\mathrm{Cov}(S_1, H_1) \ge 0$ and for the put option $\mathrm{Cov}(S_1, H_1) \le 0$.

where $\Delta X_1 = X_1 - X_0$, $\Delta X_2 = X_2 - X_1$, $X_1 = (1 + r\Delta t)^{-1} S_1$, and $X_2 = (1 + r\Delta t)^{-2} S_2$. We want to find $W_0, \xi_1, \xi_2 \in \mathbf{R}$ so that

$$E_0 \left(W_2 - C_2\right)^2 \tag{6}$$

is minimized. It is more convenient to use the value process and the discounted contingent claim: define

$$\begin{aligned} V_0 &= W_0, \\ V_1 &= (1 + r\Delta t)^{-1} W_1, \\ V_2 &= (1 + r\Delta t)^{-2} W_2, \\ H_2 &= (1 + r\Delta t)^{-2} C_2. \end{aligned}$$

Finding $W_0, \xi_1, \xi_2 \in \mathbf{R}$ minimizing (6) is equivalent to finding $V_0, \xi_1, \xi_2 \in \mathbf{R}$ minimizing

$$E_0 \left(V_2 - H_2\right)^2 .$$

We have that

$$\min_{V_0, \xi_1, \xi_2} E_0 \left(V_2 - H_2\right)^2 = \min_{V_0, \xi_1} E_0 \min_{\xi_2} E_1 \left(V_2 - H_2\right)^2 .$$

Since

$$E_1 \left(V_2 - H_2\right)^2 = E_1 (V_1 - H_2)^2 + 2\xi_2 E_1[(V_1 - H_2)\Delta X_2] + \xi_2^2 E_1 \Delta X_2^2,$$

the minimizer over $\xi_2 \in \mathbf{R}$ is

$$\xi_2 = \frac{E_1[(H_2 - V_1)\Delta X_2]}{E_1 \Delta X_2^2}. \tag{7}$$

We have that

$$\begin{aligned} &\min_{\xi_2} E_1 \left(V_2 - H_2\right)^2 \\ &= E_1 (V_1 - H_2)^2 - \frac{(E_1[(H_2 - V_1)\Delta X_2])^2}{E_1 \Delta X_2^2} \\ &= k_1 V_1^2 - 2 V_1 k_1 H_1 + k_1 H_1^2 - k_1 H_1^2 + E_1 H_2^2 - \frac{[E_1(H_2 \Delta X_2)]^2}{E_1 \Delta X_2^2} \\ &= k_1 (V_1 - H_1)^2 + \epsilon_1^2, \end{aligned} \tag{8}$$

where we denote

$$k_1 = 1 - \frac{[E_1(\Delta X_2)]^2}{E_1(\Delta X_2^2)}, \qquad H_1 = \frac{1}{k_1} \left[E_1 H_2 - \frac{E_1(\Delta X_2) E_1(\Delta X_2 H_2)}{E_1(\Delta X_2^2)} \right]$$

and

$$\epsilon_1^2 = E_1 H_2^2 - k_1 H_1^2 - \frac{[E_1(H_2 \Delta X_2)]^2}{E_1 \Delta X_2^2}.$$

It holds that

$$\min_{\xi_1,\xi_2} E_0 \left(V_2 - H_2\right)^2 = \min_{\xi_1} E_0 \left[k_1 (V_1 - H_1)^2\right] + \epsilon_1^2.$$

Similar calculations which lead to (7) show that the minimizer over $\xi_1 \in \mathbf{R}$ is

$$\xi_1 = \frac{E_0[(H_1 - V_0) k_1 \Delta X_1]}{E_0(k_1 \Delta X_1^2)}. \tag{9}$$

Finally, we have to find V_0 minimizing

$$\min_{\xi_1} E_0 \left[k_1 (V_1 - H_1)^2\right].$$

The minimizer over $V_0 \in \mathbf{R}$ is

$$H_0 = \frac{1}{k_0}\left[E_0(k_1 H_1) - \frac{E_0(k_1 \Delta X_1) E_0(k_1 \Delta X_1 H_1)}{E_0(k_1 \Delta X_1^2)}\right], \tag{10}$$

where

$$k_0 = E_0 k_1 - \frac{[E_0(k_1 \Delta X_1)]^2}{E_0(k_1 \Delta X_1^2)}.$$

Indeed, similar calculations which lead to (8) show that

$$\min_{\xi_1} E_0 \left[k_1 (V_1 - H_1)^2\right] = k_0 (V_0 - H_0)^2 + \epsilon_0^2,$$

where

$$\epsilon_0^2 = E_0(k_1 H_1^2) - k_0 H_0^2 - \frac{[E_0(k_1 H_1 \Delta X_1)]^2}{E_0(k_1 \Delta X_1^2)}.$$

We have deduced the following result.

Proposition 1 *In the two period model the fair price at time 0 is H_0 given in (10) and the optimal hedging coefficient is ξ_1 given in (9), when we define the fair price and the optimality of the hedging coefficient by the mean squared error.*

1.2.3 Approximate Pricing by Local Hedging in Two Period Model

The option price C_2 is determined at time point 2 and we can first approximate random variable C_2 with $a_2 + b_2 S_2$, where $a_2, b_2 \in \mathbf{R}$. We find a_2 and b_2 minimizing

$$E_1(a_2 + b_2 S_2 - C_2)^2,$$

where the expectation is taken conditional on the information available at time 1. Similarly as in the single period hedging we get

$$b_2 = \frac{\mathrm{Cov}_1(S_2, C_2)}{\mathrm{Var}_1(S_2)}, \qquad a_2 = E_1 C_2 - b_2 E_1 S_2.$$

Note that now the expectations, variances, and covariances are conditional on the information available at time 1. Denote by C_1 the discounted value to time point 1 of the random variable $a_2 + b_2 S_2$:

$$\begin{aligned} C_1 &= (1 + r\Delta t)^{-1} a_2 + b_2 S_1 \\ &= (1 + r\Delta t)^{-1} E_1 C_2 + \frac{\mathrm{Cov}_1(S_2, C_2)}{\mathrm{Var}_1(S_2)} \big(S_1 - (1 + r\Delta t)^{-1} E_1 S_2\big). \end{aligned}$$

The next step is to approximate C_1 by random variable $a_1 + b_1 S_1$, where $a_1, b_1 \in \mathbf{R}$. We find a_1 and b_1 minimizing

$$E_0(a_1 + b_1 S_1 - C_1)^2.$$

We get

$$b_1 = \frac{\mathrm{Cov}_0(S_1, C_1)}{\mathrm{Var}_0(S_1)}, \qquad a_1 = E_0 C_1 - b_1 E_0 S_1.$$

The price C_0 is

$$\begin{aligned} C_0 &= (1 + r\Delta t)^{-1} a_1 + b_1 S_0 \\ &= (1 + r\Delta t)^{-1} E_0 C_1 + \frac{\mathrm{Cov}_0(S_1, C_1)}{\mathrm{Var}_0(S_1)} \big(S_0 - (1 + r\Delta t)^{-1} E_0 S_1\big). \end{aligned}$$

The hedging coefficient is

$$\xi_1 = b_1 = \frac{\mathrm{Cov}_0(S_1, C_1)}{\mathrm{Var}_0(S_1)}.$$

1.3 Multiperiod Model

We consider the multiperiod model with time steps $t = 0, \ldots, T$. Let C_T be the value of the option at the expiration.

1.3.1 Exact Pricing in the Multiperiod Model

Here we follow Černý (2004, Section 13.4).

Theorem 2 *Let $H_T = (1 + r\Delta t)^{-T} C_T$ and let H_t be defined recursively for $t = 0, \ldots, T-1$ by*

$$H_t = \frac{1}{k_t}\left[E_t(k_{t+1}H_{t+1}) - \frac{E_t(k_{t+1}\Delta X_{t+1})E_t(k_{t+1}\Delta X_{t+1}H_{t+1})}{E_t[k_{t+1}\Delta X_{t+1}^2]}\right],$$

where $k_T = 1$ and

$$k_t = \left(E_t k_{t+1} - \frac{(E_t[k_{t+1}\Delta X_{t+1}])^2}{E_t[k_{t+1}\Delta X_{t+1}^2]}\right).$$

Then the fair price in the mean squared error sense is H_0 and the optimal hedging coefficient is

$$\xi_1 = \frac{E_0[k_1(H_1 - H_0)\Delta X_1]}{E_0[k_1 \Delta X_1^2]}.$$

1.3.2 Approximate Pricing by Local Hedging in the Multiperiod Model

The option price C_T is known and fixed at the expiration date T. For $t = T, \ldots, 1$ we define

$$C_{t-1} = (1 + r\Delta t)^{-1} a_t + b_t S_{t-1}, \tag{11}$$

where

$$b_t = \frac{\mathrm{Cov}_{t-1}(S_t, C_t)}{\mathrm{Var}_{t-1}(S_t)}, \qquad a_t = E_{t-1}C_t - b_t E_{t-1}S_t. \tag{12}$$

That is, $a_t, b_t \in \mathbf{R}$ are chosen so that

$$E_{t-1}(a_t + b_t S_t - C_t)^2$$

is minimized. We can write the price C_{t-1} as

$$C_{t-1} = (1 + r\Delta t)^{-1} E_{t-1}C_t + \frac{\mathrm{Cov}_{t-1}(S_t, C_t)}{\mathrm{Var}_{t-1}(S_t)}\left(S_{t-1} - (1 + r\Delta t)^{-1} E_{t-1}S_t\right).$$

The backward calculation leads to the price C_0 and to the hedging coefficient b_1 that can be used at time $t = 0$ of writing the option to price and hedge the option.

References

Černý, A. (2004), *Mathematical Techniques in Finance: Tools for Incomplete Markets*, Princeton University Press.