Lecture Notes 6

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1 Return of a Portfolio

Portfolio selection starts with choosing the portfolio components. The portfolio components can be stocks, bonds, commodities, currencies, or other financial assets. The risk free rate can also be included in the portfolio.[1] Let us assume that we have N portfolio components and let

$$S_t = (S_t^1, \ldots, S_t^N)'$$

be the vector of prices of portfolio components at time t. Each price satisfies $0 \le S_t^i < \infty$. A portfolio is described by giving the portfolio weights

$$b_t = (b_t^1, \ldots, b_t^N)'$$

that satisfy

$$\sum_{i=1}^{N} b_t^i = 1 \tag{1}$$

at each time t. The number b_t^i is equal to the proportion of the total wealth invested in asset i at time t.

Let W_t be the total wealth at time t available for investment and let ξ_{t+1}^i the number of assets i, $i = 1, \ldots, N$, bought at time t. We assume that ξ_{t+1}^i can be any real number and not just an integer. Random variable ξ_{t+1} is chosen at time t, using at time t available information, and its value at time $t+1$ is known at time t: it is said that ξ_{t+1} is a predictable random variable. We allocate the total wealth among the portfolio components:

$$W_t = \sum_{i=1}^{N} \xi_{t+1}^i S_t^i. \tag{2}$$

[1] The risk free rate is different depending on the investment horizon. For the one day horizon the risk free rate could be the Eonia rate or the rate of a bank account, and for the one month horizon the risk free rate could be the rate of a one month governement bond.

This self-financing condition states that no wealth is reserved for consumption and no wealth is inserted outside to the portfolio. The portfolio weights are

$$b_t^i = \frac{\xi_{t+1}^i S_t^i}{W_t}.$$

Thus, condition (2) is equivalent to condition (1). At time $t+1$ the wealth is

$$W_{t+1} = \sum_{i=1}^{N} \xi_{t+1}^i S_{t+1}^i.$$

Next wealth W_{t+1} is allocated among the portfolio components choosing the numbers of assets ξ_{t+2}^i, $i = 1, \ldots, N$. The gross return of the portfolio is

$$R_{t+1}^p = \frac{W_{t+1}}{W_t} = \sum_{i=1}^{N} \frac{\xi_{t+1}^i S_t^i}{W_t} \frac{S_{t+1}^i}{S_t^i} = \sum_{i=1}^{N} b_t^i R_{t+1}^i = b_t' R_{t+1}, \qquad (3)$$

where $R_{t+1} = (R_{t+1}^1, \ldots, R_{t+1}^N)'$ is the column vector of the gross returns of the portfolio components, where the gross return of a single portfolio component is

$$R_{t+1}^i = \frac{S_{t+1}^i}{S_t^i}.$$

Thus, the return of the portfolio is obtained as a weighted average of the returns of the portfolio components.

As an example, consider the case where the portfolio components are two stocks whose returns are $R_t^{(i)} = S_t^{(i)}/S_{t-1}^{(i)}$, $i = 1, 2$. Then the one period return of the portfolio is

$$R_t^p(b) = bR_t^1 + (1-b)R_t^2.$$

2 Portfolio Types

A portfolio is selected from a collection of portfolios which can be described by giving the collection of possible portfolio weights. The most general collection of portfolio weights consists of all weights satisfying the constraint (1):

$$B = \left\{ (b_t^1, \ldots, b_t^N) : \sum_{i=1}^{N} b_t^i = 1 \right\}.$$

We can impose various restrictions on portfolio weights and obtain smaller collections of weights. For exaple, we can allow leveraging but forbid shorting of stocks, or we can restrict ourselves to long only portfolios.

2.1 Long Only Portfolios

In a long only portfolio borrowing and short selling are excluded. In the case of long only portfolios the portfolio weights are nonnegative. Thus the weights satisfy

$$b_t^j \geq 0 \text{ for } j = 1, \ldots, N.$$

The nonnegativity constraint together with the condition $\sum_{j=1}^N b_t^j = 1$ imply that

$$0 \leq b_t^j \leq 1$$

for $j = 1, \ldots, N$.

2.2 Leveraged Portfolios

A portfolio allowing leveraging but forbidding short selling is such that the weight of the risk free rate can be negative but the weights of the other assets are nonnegative. In a leveraged portfolio it is allowed to borrow money and invest borrowed money to stocks or other assets. Borrowing money is interpreted as shorting the risk free rate. Let S_t^1 be the bank account. The portfolio vectors of a leveraged portfolio satisfy, in addition to the constraint $\sum_{j=1}^N b_t^j = 1$, the additional constraint

$$b_t^j \geq 0 \text{ for } j = 2, \ldots, N.$$

We allow negative values for the portfolio weight b_t^1 of the bank account, but the other portfolio weights b_t^j, $j = 2, \ldots, N$ are nonnegative.

2.3 Restrictions on Short Selling

In practice investors have a constraint on the amount of short selling. It is natural to make a constraint on the amount of short selling by requiring that the portfolio weights satisfy

$$\sum_{j=1}^N |b_t^j| \leq L, \tag{4}$$

where $L \geq 1$. Under the constraint $\sum_{j=1}^N b_t^j = 1$, the constraint (4) is equivalent to any of the following two constraints:

$$\sum_{j=1}^N (b_t^j)_- \leq \frac{L-1}{2}, \qquad \sum_{j=1}^N (b_t^j)_+ \leq \frac{L+1}{2},$$

where we denote by $(b)_+ = \max\{0, b\}$ the positive part of $b \in \mathbf{R}$ and by $(b)_- = -\min\{0, b\}$ the negative part of b.[2] Thus, $C = (L-1)/2$ is such factor that we are allowed to short sell C times the current wealth.

3 Wealth Process and Investment Strategy

The gross return of a portfolio was given in (3) as

$$R^p_{t+1} = b'_t R_{t+1},$$

where b_t is the vector of the weights of the portfolio components and R_{t+1} is the vector of gross returns of the portfolio components. The gross return of the portfolio is the factor by which the capital invested in the portfolio increases during the trading period. Let us assume to have initial wealth $W_{t_0} > 0$ which we distribute among the portfolio components. The wealth process $W_{t_0}, W_{t_0+1}, \ldots$ of a portfolio is defined using the recursive formula

$$W_{t+1} = W_t \cdot b'_t R_{t+1}, \qquad t = t_0, t_0 + 1, \ldots. \tag{5}$$

This equation holds because the strategy is self-financing and there is no consumption; wealth W_{t+1} is obtained from wealth W_t only through the changes in asset prices and through the changes in wealth allocation. After $(t - t_0)$-trading periods we obtain the wealth

$$W_t = W_{t_0} \prod_{i=t_0}^{t-1} b'_i R_{i+1}, \tag{6}$$

where the initial wealth W_{t_0} is a fixed number, but $W_{t_0+1}, W_{t_0+2}, \ldots$ are real valued random variables.

When the sequence b_t, $t = t_0, t_0 + 1, \ldots$, of portfolio vectors is constant, not changing with t, then we call the portfolios "constant weight portfolios". Note that when using the constant weight portfolio strategy there is a need to

[2] We have $b = (b)_+ - (b)_-$ and $|b| = (b)_+ + (b)_-$. Thus,

$$\sum_{j=1}^N b_t^j = 1 \Leftrightarrow \sum_{j=1}^N (b_t^j)_+ = 1 + \sum_{j=1}^N (b_t^j)_-.$$

Then,

$$\sum_{j=1}^N |b_t^j| = \sum_{j=1}^N (b_t^j)_+ + \sum_{j=1}^N (b_t^j)_- \le L \Leftrightarrow \sum_{j=1}^N (b_t^j)_- \le \frac{L-1}{2}.$$

make a rebalancing at each period because the prices of the portfolio components are changing and to keep the weights constant we have to decrease the weight of those assets whose price has increased and to increase the weights of those assets whose price has declined. In this sense a constant weight portfolio strategy is a counter trend strategy.

A varying weight portfolio strategy will typically choose the weights using available relevant information. We assume that the relevant information is expressed with the state vector $Z_t \in \mathbf{R}^d$. The vector $b_t \in \mathbf{R}^N$ of weights is obtained with a function

$$w : \mathbf{R}^d \to \mathbf{R}^N$$

and we have

$$b_t = w(Z_t).$$

More generally, the function w may be changing with time and the definition of the relevant information Z_t may be changing with time.

4 Single and Multiperiod Portfolio Selection

The single period return of the portfolio is defined in (3) as

$$R^p_{t+1} = \frac{W_{t+1}}{W_t} = b_t' R_{t+1}, \tag{7}$$

where b_t is the vector of portfolio weights and R_t is the vector of the gross returns of the portfolio components, where the gross return of a single portfolio component is $R^i_t = S^i_t / S^i_{t-1}$. The multiperiod return of the portfolio is obtained from (6) as

$$R^p_T = \frac{W_T}{W_{t_0}} = \prod_{i=t_0}^{T-1} b_i' R_{i+1}. \tag{8}$$

In the single period case we have to choose the weights b_t so that the next period return R^p_{t+1} is optimized. In the multiperiod case we have to choose the sequence $b_{t_0}, \ldots, b_{T-1}$ so that the return R^p_T at time T is optimized.