Lecture Notes 7

Jussi Klemelä

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1 Comparison of Distributions

In order to choose the portfolio weights so that the return of the portfolio is optimized we need to define when a return distribution is better than another return distribution. Also, to measure the performance of asset managers we need to define whether a return distribution generated by an asset manager is better than the return distribution generated by an other asset manager. Alternatively, instead of considering return distributions we can consider distributions of wealth.

Figure 1 illustrates the comparison of distributions. Panel (a) shows two densities of distributions which are easy to compare; the densities have the same shape but the other dominates the other, because its mode is at 1.2 wheras the mode of the other density is at 1.05. Panel (b) shows two densities which cannot be compared straightforwardly; the mode of the other is at 1.2 but its variance is larger, whereas the mode of the other is at 1.05 but its variance is smaller.

Figure 1 shows an example of stochastic dominance. We say that the distribution of $U^{(1)}$ stochastically dominates the distribution of $U^{(2)}$ if $F_2(u) \geq F_1(u)$ for all $u \in \mathbf{R}$, where F_1 is the distribution function of $U^{(1)}$ and F_2 is the distribution function of $U^{(2)}$. [1] If the distribution of $U^{(1)}$ stochastically dominates the distribution of $U^{(2)}$ then $P(U^{(1)} \geq u) \geq P(U^{(2)} \geq u)$ for all $u \in \mathbf{R}$.

[1]Stochastic dominance occurs if and only if the dominant distribution has a higher expected utility for all increasing and continuously differentiable utility functions. Stochastic dominance is also called first order stochastic dominance to distinguish it from the second order stochastic dominance.

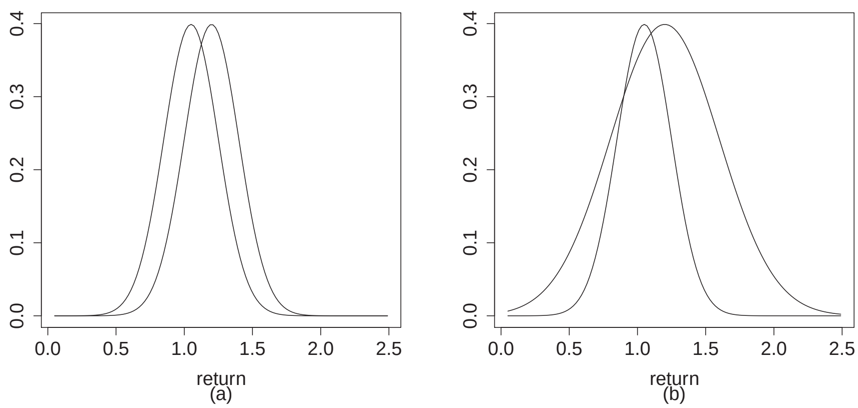

Figure 1: *Comparison of distributions.* (a) Two return densities which are easy to compare. (b) Two return densities which are difficult to compare.

2 Expected Utility

We can order distributions according to the value of the expected utility. The expected utility can be calculated either from the returns or from the wealth. The expected utility calculated from the wealth is

$$Eu(W_t),$$

where W_t is the wealth (in Euros, Dollars, etc.), and $u : \mathbf{R} \to \mathbf{R}$ is a utility function. The negative wealth means that more is borrowed than owned. The expected utility calculated from the gross returns is

$$Eu(R_t),$$

where $u : (0, \infty) \to \mathbf{R}$ is a utility function and $R_t = W_t/W_{t-1}$ is the gross return calculated from wealth W_t. We can assume that the gross return R_t is always nonnegative. It is natural to define $u(0) = -\infty$, because the gross return of zero means bankruptcy.

2.1 Utility Functions

It is natural to require that a utility function is strictly increasing and strictly concave. A concave function is such that the rate of increase decreases; see the definition in (3). A utility function should be increasing because people prefer a larger wealth to a lesser wealth. A utility function should be concave because when the wealth increases, the value of additional wealth declines. The power utility functions are defined as

$$u(t) = \begin{cases} \frac{t^{1-\gamma}}{1-\gamma}, & \text{if } \gamma > 1, \\ \log t, & \text{if } \gamma = 1, \end{cases} \qquad t > 0. \tag{1}$$

Note that $u'(t) = t^{-\gamma}$ for $\gamma > 1$ and $\partial/\partial t \log t = t^{-1}$, which can be used to explain why the logarithmic function is obtained as a limit when $\gamma \downarrow 1$. The power utility functions are constant relative risk aversion (CRRA) utility functions, as defined in (5). The exponential utility functions are

$$u(t) = 1 - e^{-\alpha t}, \qquad t \in \mathbf{R}, \tag{2}$$

where $\alpha > 0$. The exponential utility functions are constant absolute risk aversion (CARA) utility functions, as defined in (4). Note that the power utility functions are defined on $(0, \infty)$ but the exponential utility functions are defined on the whole real line and can thus be applied in the case of negative wealth.[2]

Figure 2 plots normalized utility functions with different risk aversion parameters. Panel (a) shows power utility functions (1) and panel (b) shows exponential utility functions (2). The normalized utility functions $\tilde{u}$ are defined by

$$\tilde{u}(t) = \frac{u(t) - u(1)}{u(2) - u(1)}.$$

The normalization is such that $\tilde{u}(1) = 0$ and $\tilde{u}(2) = 1$. Note that the ordering of the distributions is not affected by linear transformations $au(t) + c$, $a > 0$, $c \in \mathbf{R}$, because

$$E[au(R_t) + c] = aEu(R_t) + c.$$

Figure 2 shows that larger values of γ or α are used when one is more risk averse, because the curvature of the utility functions increases when γ or α are increased.

Figure 3 shows contour plots of functions $(\sigma, \mu) \mapsto Eu(U)$, where U follows distribution $U \sim 1 + r$, where $r \sim N(\mu_0, \sigma_0^2)$, where $\mu_0 = \mu/250$, $\sigma_0 = \sigma/\sqrt{250}$. In panel (a) the utility function is logarithmic $u(x) = \log x$ and in panel (b) $u(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma = 5$.[3] The expected utility is maximized when the mean is high and the standard deviation is low, which

[2] The utility functions $u(t) = I_{[a,\infty)}(t)$, $t \in \mathbf{R}$, where $a > 0$, are used when one wants to choose a portfolio maximizing the probability of reaching the given amount of capital, although they are not concave functions.

[3] I panel (a) we have multiplied the values of $E(u(U))$ with 1000 and in panel (b) with 10.

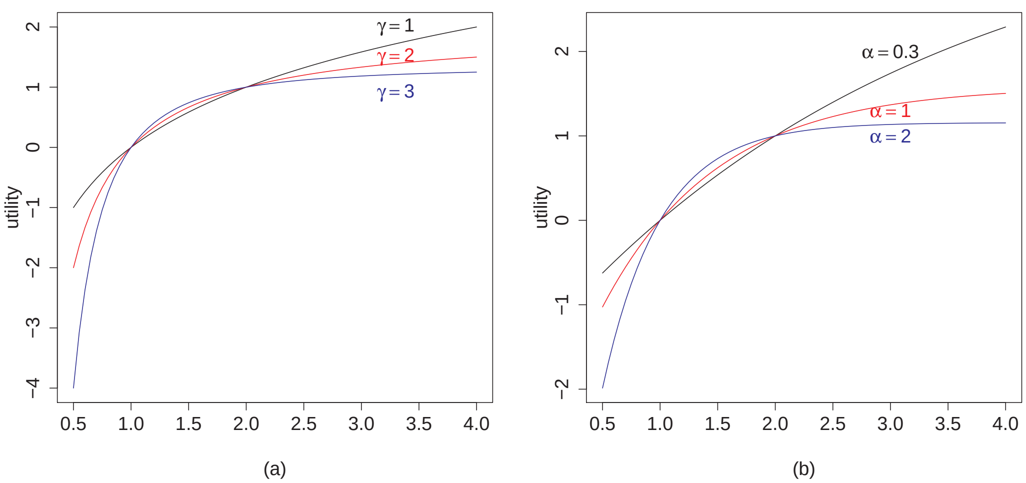

Figure 2: *Utility functions.* (a) Power utility functions (1) for risk aversion values $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$ and (b) exponential utility functions (2) for risk aversion values $\alpha = 0.3$, $\alpha = 1$, and $\alpha = 2$.

happens in the upper left corner. We see that for the logarithmic utility the expected utility is determined by the expectation, but increasing the risk aversion to $\gamma = 5$ makes the expected utility sensitive both to mean and to standard deviation. When risk aversion is increased more, then the expected utility becomes sensitive only to standard deviation.

2.2 Risk Aversion

The concavity of the utility function is a consequence of risk aversion. A strictly increasing function $f : \mathbf{R} \to \mathbf{R}$ satisfies $f'(x) > 0$ for all $x \in \mathbf{R}$ and a strictly concve function satisfies $f''(x) < 0$ for all $x \in \mathbf{R}$, when the function is two times differentiable. Concavity can be defined also in the case where the function is not two times differentiable. A function $f : \mathbf{R} \to \mathbf{R}$ is strictly concave, when

$$f(px_1 + (1-p)x_2) > pf(x_1) + (1-p)f(x_2) \tag{3}$$

for all $0 \le p \le 1$ and for all $x_1, x_2 \in \mathbf{R}$. Let us consider the distribution of return R_t, where

$$P(R_t = 1 - \alpha) = p, \qquad P(R_t = 1 + \alpha) = 1 - p$$

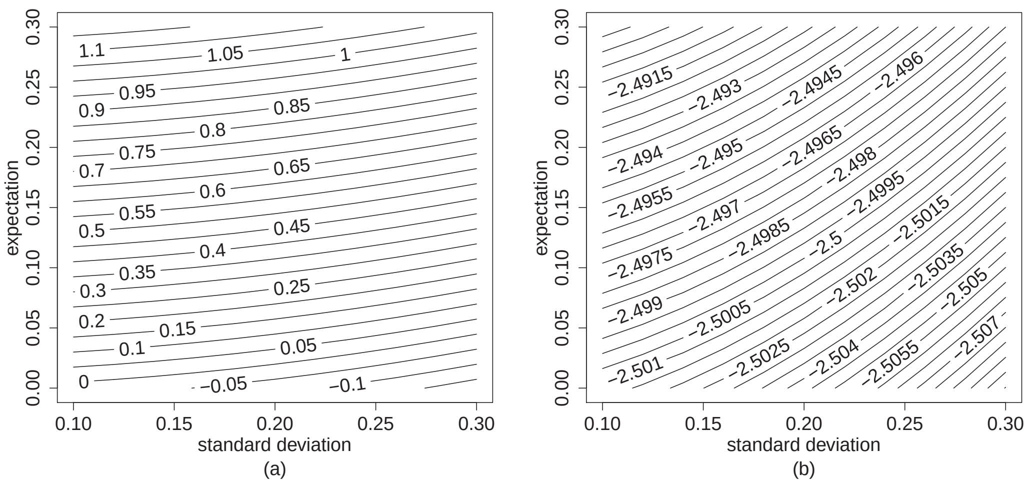

Figure 3: *Expected utility as a funcrion of mean and standard deviation.* We show contour plots of functions $(\sigma, mu) \mapsto Eu(U)$, where U follows a normal distribution. (a) $u(x) = \log x$; (b) $u(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma = 5$.

for some $a > 0$ and $0 \le p \le 1$. Then, for a concave utility function $u : (0, \infty) \to \mathbf{R}$,

$$\begin{aligned} Eu(R_t) &= pu(1-\alpha) + (1-p)u(1+\alpha) \\ &< u[p(1-\alpha) + (1-p)(1+\alpha)] \\ &= u(1+\alpha(1-2p)). \end{aligned}$$

Thus one would prefer always the certainly received amount $1+\alpha(1-2p)$ to the lottery. In particular, in the case $p = 1/2$, one would prefer to preserve the current wealth to the lottery with equal probabilities $1/2$ of winning and loosing the amount α. The number

$$u^{-1}(Eu(R_t))$$

is called the certainty equivalent, since this is the minimum amount of wealth, guaranteed preservation of which allows the investor to decline the proposed game.

We can classify utility functions using measures of risk aversion. The coefficient of absolute risk aversion of utility function u at point t is defined as

$$-\frac{u''(t)}{u'(t)}. \tag{4}$$

Utility functions with constant absolute risk aversion are called CARA utility functions. For example, the exponential utility functions, defined in (2), are CARA utility functions and have the coefficient of absolute risk aversion α, wheras the power utility functions, defined in (1), are not CARA utility functions because they have the coefficient of absolute risk aversion γt^{-1}. When an investor whose wealth is 100 is willing to risk 50, and after reaching wealth 1000, is still willing to risk 50, then the investor has constant absolute risk aversion. Most investors have decreasing absolute risk aversion. The coefficient of relative risk aversion of utility funcrion u at point t is defined as

$$-t\,\frac{u''(t)}{u'(t)}. \tag{5}$$

Utility functions with constant relative risk aversion are called CRRA utility functions. For example, the power utility functions are CRRA utility functions and have the coefficient of relative risk aversion γ, wheras the exponential utility functions are not CRRA utility functions because they have the coefficient of relative risk aversion αt. When an investor whose wealth is 100 is willing to risk 50, and after reaching wealth 1000, is willing to risk 500, then the investor has constant relative risk aversion. Investors are usually assumed to have constant relative risk aversion.

3 Mean-Variance Preferences

We can order distributions according to

$$EY - \frac{\gamma}{2}\,\mathrm{Var}(Y),$$

where $\gamma \geq 0$ is the coefficient of absolute risk aversion. Parameter γ measures the investor's absolute risk aversion as defined in (4).

4 Sharpe Ratio

The Sharpe ratio is defined as the expected excess return divided by the standard deviation of the excess return:

$$\frac{E(R_t - r_t)}{\mathrm{sd}(R_t - r_t)},$$

where R_t is the return of a portfolio, and r_t is the return of a risk free investment. The portfolio returns and the risk free returns can be gross returns or net returns. The return period can be one day, one month, or one year, for example. The annualized Sharpe ratio is defined as

$$(\Delta t)^{-1/2} \frac{E(R_t - r_t)}{\mathrm{sd}(R_t - r_t)},$$

where $\Delta t = 1/250$ for daily returns, $\Delta t = 1/12$ for monthly returns, and so on.

An estimator of the Sharpe ratio is obtained from historical returns $R_1, \ldots, R_T$ and from historical risk free rates $r_1, \ldots, r_T$ as

$$(\Delta t)^{-1/2} \frac{T^{-1} \sum_{t=1}^{T} (R_t - r_t)}{\widehat{\mathrm{sd}}(R_t - r_t)},$$

where $\widehat{\mathrm{sd}}(R_t - r_t)$ is the sample standard deviation:

$$\widehat{\mathrm{sd}}(R_t - r_t) = \left(\frac{1}{T} \sum_{t=1}^{T} (R_t - r_t) - \frac{1}{T} \sum_{t=1}^{T} (R_t - r_t)^2 \right)^{1/2}.$$