Lecture Notes 8

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November 27, 2014

1 Markowitz Portfolios

Portfolio choice with mean-variance preferences was proposed by Markowitz (1952) and Markowitz (1959). Markowitz appraoch can be used in the single period portfolio selection. Let R^p_{t+1} be the portfolio return. There exists three ways to choose the portfolio:

1. Maximize the variance penalized expected return

$$ER^p_{t+1} - \frac{\gamma}{2} \operatorname{Var}\left(R^p_{t+1}\right).$$

   where $\gamma \geq 0$ is the risk aversion coefficient.

2. Minimize the variance $\operatorname{Var}(R^p_{t+1})$ under a minimal requirement for the expected return: $ER^p_{t+1} \geq \mu_0$.

3. Maximize the expected return ER^p_{t+1} under a condition that the variance is not too large: $\operatorname{Var}(R^p_{t+1}) \leq \sigma_0^2$.

Section 2 considers the maximization of the variance penalized expected return. Let R_{t+1} be the vector of returns of N risky assets and let $b = (b^1, \ldots, b^N)'$ be a vector of N portfolio weights. The portolio return is

$$R^p_{t+1} = b' R_{t+1}.$$

Let us denote $\mu = ER_{t+1}$ and $\Sigma = \operatorname{Cov}(R_{t+1})$. The vector of expected returns is a column vector of N elements and the covariance matrix Σ is a $N \times N$ matrix. Now,

$$E(b' R_{t+1}) = b' \mu, \qquad \operatorname{Var}(b' R_{t+1}) = b' \Sigma b.$$

The maximization of the variance penalized expected return chooses the weight vector b as maximizing

$$b'\mu - \frac{\gamma}{2} b'\Sigma b \text{ under } b'1_N = 1, \tag{1}$$

where 1_N is the vector of length N whose all elements are equal to one, so that the constraint is

$$b'1_N = \sum_{i=1}^{N} b^i = 1.$$

We consider also adding the risk free rate as a possible investment. Let the return of the risk free investment be μ_r. Let us invest the proportion $1-b'1_N$ to the risk free investment. Then the portfolio return is

$$R^p_{t+1} = b'R_{t+1} + (1 - b'1_N)\mu_r.$$

We choose the weight vector b as maximizing

$$b'\mu + (1 - b'1_N)\mu_r - \frac{\gamma}{2} b'\Sigma b \text{ under } b'1_N = 1. \tag{2}$$

Section 3 considers minimization of the variance under a minimal requirement for the expeced return. The method chooses the weight vector as minimizing

$$b'\Sigma b \text{ under } b'\mu = \mu_0, b'1_N = 1, \tag{3}$$

where $\mu_0 \in \mathbf{R}$, and we should choose $\mu_0 \geq \mu_r$. When the risk free rate is included, then the method chooses the weight vector as minimizing

$$b'\Sigma b \text{ under } b'\mu + (1 - b'1_N)\mu_r = \mu_0. \tag{4}$$

2 Variance Penalized Expected Return

2.1 Variance Penalization with the Risk Free Rate

Let us consider the maximization of the variance penalized expected return (2) when the risk free rate is included. We solve first the case of one risky asset, then the case of two risky assets, and finally the case of N risky assets.

2.1.1 One Risky Asset and the Risk Free Rate

Let us invest the proportion b to a stock and $1-b$ to the risk free rate whose gross return is $\mu_r > 0$. Now the gross return of the portfolio is

$$R^p_{t+1} = bR^s_{t+1} + (1 - b)\mu_r,$$

where R^s_{t+1} is the return of the stock. Let the expected return of the stock be $ER^s_{t+1} = \mu$ and the variance $\mathrm{Var}(R^s_{t+1}) = \sigma^2$. Then,

$$ER^p_{t+1} - \frac{\gamma}{2}\,\mathrm{Var}(R^p_{t+1}) = \mu_r + b(\mu - \mu_r) - \frac{\gamma}{2}\,b^2\sigma^2.$$

Setting the derivative with respect to b to zero and solving for b gives

$$b = \frac{1}{\gamma}\,\frac{\mu - \mu_r}{\sigma^2}.$$

2.1.2 Two Risky Assets and the Risk Free Rate

Let us have two stocks and the risk free rate and let us put the proportion b_1 to the first stock, proportion b_2 to the second stock, and proportion $1-b_1-b_2$ to the risk free rate. Now the portfolio return is

$$R^p_{t+1} = b_1 R^1_{t+1} + b_2 R^2_{t+1} + (1 - b_1 - b_2)\mu_r,$$

where R^1_{t+1} is the return of the first stock and R^2_{t+1} is the return of the second stock. Let the expected returns of the stocks be $ER^1_{t+1} = \mu_1$, $ER^2_{t+1} = \mu_2$ and let the variances of the returns be $\mathrm{Var}(R^1_{t+1}) = \sigma_1^2$, $\mathrm{Var}(R^2_{t+1}) = \sigma_2^2$. Denote the covariance of the returns by $\mathrm{Cov}(R^1_{t+1}, R^2_{t+1}) = \sigma_{12}$. We have

$$\begin{aligned} & ER^p_{t+1} - \frac{\gamma}{2}\,\mathrm{Var}(R^p_{t+1}) \\ = \;& b_1\mu_1 + b_2\mu_2 + (1 - b_1 - b_2)\mu_r - \frac{\gamma}{2}\,(b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}). \end{aligned}$$

Setting derivatives with respect to b_1 and b_2 to zero gives

$$\begin{cases} \mu_1 - \mu_r - \gamma\sigma_1^2 b_1 - \gamma\sigma_{12} b_2 = 0, \\ \mu_2 - \mu_r - \gamma\sigma_2^2 b_2 - \gamma\sigma_{12} b_1 = 0. \end{cases}$$

Thus,[1]

$$b_1 = \frac{1}{\gamma}\,\frac{(\mu_1 - \mu_r)\sigma_2^2 - \sigma_{12}(\mu_2 - \mu_r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$$

and

$$b_2 = \frac{1}{\gamma}\,\frac{(\mu_2 - \mu_r)\sigma_1^2 - \sigma_{12}(\mu_1 - \mu_r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}.$$

[1] Solving b_2 gives

$$b_2 = \frac{\mu_2 - \mu_r - \gamma\sigma_{12} b_1}{\gamma\sigma_2^2}.$$

This leads to

$$\mu_1 - \mu_r - \frac{\sigma_{12}}{\sigma_2^2}\,(\mu_2 - \mu_r) + b_1\gamma\left(\frac{\sigma_{12}^2}{\sigma_2^2} - \sigma_1^2\right) = 0.$$

2.1.3 Sevaral Risky Assets and the Risk Free Rate

Let us consider the case of N risky assets and assume that the proportion $1 - b'1_N$ is invested in the risk-free asset. Let us maximize

$$b'\mu + (1 - b'1_N)\mu_r - \frac{\gamma}{2}\, b'\Sigma b$$

where $\mu_r > 0$ is the return of the risk-free asset. Derivating with respect to b and setting the partial derivatives to zero we get

$$\mu - 1_N\mu_r - \gamma\Sigma b = 0.$$

Thus,

$$b = \gamma^{-1}\Sigma^{-1}(\mu - \mu_r 1_N).$$

2.2 Variance Penalization without the Risk Free Rate

Let us consider the maximization of the variance penalized expected return (1) when the risk free rate is excluded. We solve first the case of two risky assets and then the case of N risky assets.

2.2.1 Two Risky Assets

Let us have two stocks and put the proportion $1 - b$ to the first stock and proportion b to the second stock. Now

$$R^p_{t+1} = (1 - b)R^1_{t+1} + bR^2_{t+1}.$$

Let the expected returns of the stocks be $ER^1_{t+1} = \mu_1$, $ER^2_{t+1} = \mu_2$ and let the variances of the returns be $\mathrm{Var}(R^1_{t+1}) = \sigma_1^2$, $\mathrm{Var}(R^2_{t+1}) = \sigma_2^2$. Denote the covariance of the returns by $\mathrm{Cov}(R^1_{t+1}, R^2_{t+1}) = \sigma_{12}$. We have

$$\begin{aligned} & ER^p_{t+1} - \frac{\gamma}{2}\,\mathrm{Var}(R^p_{t+1}) \\ &= \mu_1 + b(\mu_2 - \mu_1) - \frac{\gamma}{2}\left[(1-b)^2\sigma_1^2 + b^2\sigma_2^2 + 2(1-b)b\sigma_{12}\right] \\ &= \mu_1 - \frac{\gamma}{2}\sigma_1^2 + b\left[\mu_2 - \mu_1 - \gamma(\sigma_{12} - \sigma_1^2)\right] - b^2\,\frac{\gamma}{2}\,(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}). \end{aligned}$$

Setting the derivative with respect to b to zero and solving for b gives

$$b = \frac{1}{\gamma}\,\frac{\mu_2 - \mu_1 - \gamma(\sigma_{12} - \sigma_1^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}. \tag{5}$$

Note that the maximizer b_{rest} under the restriction that $b \in [0, 1]$ is obtained by projecting the unrestricted solution:

$$b_{rest} = \min\{\max\{0, b\}, 1\},$$

where b is given in (5).

2.2.2 Several Risky Assets

Let us maximize

$$b'\mu - \frac{\gamma}{2} b'\Sigma b$$

under the constraint $b'1_N = 1$. We use the method of Lagrange multipliers and maximize

$$b'\mu - \frac{\gamma}{2} b'\Sigma b + \lambda(b'1_N - 1),$$

where $\lambda \in \mathbf{R}$ is the Lagrange multiplier. Derivating with respect to b and λ and setting the partial derivatives to zero we get

$$\begin{cases} \mu - \gamma\Sigma b + \lambda 1_N = 0, \\ b'1_N - 1 = 0. \end{cases}$$

Thus,

$$b = \gamma^{-1}\Sigma^{-1}(\mu + \lambda 1_N).$$

Let us solve λ from $b'1_N = 1$, which leads to

$$\gamma^{-1}1_N'\Sigma^{-1}\mu + \lambda\gamma^{-1}1_N'\Sigma^{-1}1_N = 1,$$

and finally

$$\lambda = \frac{1 - \gamma^{-1}1_N'\Sigma^{-1}\mu}{\gamma^{-1}1_N'\Sigma^{-1}1_N}.$$

3 Minimizing Variance under Sufficient Excpected Return

Let us consider the minimization of the variance of the portfolio under a minimal requirement for the expected return of the portfolio The minimization problem was written in (4) with the risk free rate and in (3) without the risk free rate.

3.1 Minimizing Variance with the Risk Free Rate

We consider first the case of one risky asset and the risk free investment, and then the case of N risky assets and the risk free investment.

3.1.1 One Risky Asset and the Risk Free Rate

Let us consider the case, where we have one risky asset with return R_{t+1} and a risk free investment with return μ_r. Let the expected return of the risky asset be $ER_{t+1} = \mu$ and the variance $\mathrm{Var}(R_{t+1}) = \sigma^2$. Let us invest the proportion b to the risky asset and the proportion $1-b$ to the risk free asset. The return of the portfolio is

$$R^p_{t+1} = bR_{t+1} + (1-b)\mu_r.$$

The expected return of the portfolio is

$$ER^p_{t+1} = b\mu + (1-b)\mu_r = \mu_r + b(\mu - \mu_r)$$

and the variance of the portfolio is

$$\mathrm{Var}\left(R^p_{t+1}\right) = b^2\sigma^2.$$

We want that the expected return should be at least $\mu_0 \in \mathbf{R}$. and we minimize the variance under this condition. Thus, we want to find b minimizing

$$\frac{1}{2}\, b^2\sigma^2$$

under the constraint

$$\mu_r + b(\mu - \mu_r) = \mu_0.$$

Define the Lagrange function

$$L(b,\lambda) = \frac{1}{2}\, b^2\sigma^2 + \lambda[\mu_0 - \mu_r - b(\mu - \mu_r)],$$

where $\lambda \in \mathbf{R}$ is the Lagrange multiplier. The solution of the equation

$$0 = \frac{\partial}{\partial b}\, L(b,\lambda) = \sigma^2 b - \lambda(\mu - \mu_r)$$

is

$$b = \lambda\, \frac{\mu - \mu_r}{\sigma^2}.$$

The constraint $b(\mu - \mu_r) = \mu_0 - \mu_r$ implies $\lambda = \sigma^2(\mu_0 - \mu_r)/(\mu - \mu_r)^2$. Thus, the weight of the risky investment is

$$b = \frac{\mu_0 - \mu_r}{\mu - \mu_r}.$$

When $\mu_r \le \mu_0 \le \mu$, then $0 \le b \le 1$.

3.1.2 Several Risky Assets and the Risk Free Rate

Let us consider the case of N risky assets and a risk free investment. The return vector of the risky investments is denoted by R_{t+1}, the expectation vector is μ, the covariance matrix is Σ, and the risk free return is μ_r. The proportion $1 - b'1_N$ is invested in the risk-free asset. The return of the portfolio is

$$R^p_{t+1} = b'R_{t+1} + (1 - b'1_N)\mu_r.$$

The expected return of the portfolio is

$$b'\mu + (1 - b'1_N)\mu_r.$$

Let us find b minimizing

$$\frac{1}{2}\, b'\Sigma b$$

under the constraint

$$b'\mu + (1 - b'1_N)\mu_r = \mu_0,$$

where $\mu_0 \in \mathbf{R}$ is a constant. Define the Lagrange function

$$L(b, \lambda) = \frac{1}{2}\, b'\Sigma b + \lambda[\mu_0 - b'\mu - (1 - b'1_N)\mu_r],$$

where $\lambda \in \mathbf{R}$ is the Lagrange multiplier. We solve the equation

$$0 = \frac{\partial}{\partial b}\, L(b, \lambda) = \Sigma b + \lambda(1_N\mu_r - \mu)$$

to get

$$b = \lambda\Sigma^{-1}(\mu - 1_N\mu_r).$$

The constraint can be written as

$$b'(\mu - \mu_r 1_N) = \mu_0 - \mu_r$$

which implies

$$\lambda(\mu - \mu_r 1_N)'\Sigma^{-1}(\mu - \mu_r 1_N) = \mu_0 - \mu_r$$

and

$$\lambda = \frac{\mu_0 - \mu_r}{(\mu - \mu_r 1_N)'\Sigma^{-1}(\mu - \mu_r 1_N)}.$$

Thus, the vector of the weights of the risky investments is

$$b = \frac{\mu_0 - \mu_r}{(\mu - \mu_r 1_N)'\Sigma^{-1}(\mu - \mu_r 1_N)}\, \Sigma^{-1}(\mu - \mu_r 1_N).$$

3.2 Minimizing Variance without the Risk Free Rate

Let us consider portfolios of N risky assets and exclude the risk free investment. The return vector of the risky investments is denoted by R_{t+1}. Let us denote $\mu = ER_{t+1}$ and $\Sigma = \mathrm{Cov}(R_{t+1})$. Then,

$$E(b'R_{t+1}) = b'\mu, \qquad \mathrm{Var}(b'R_{t+1}) = b'\Sigma b.$$

We minimize

$$\frac{1}{2}\, b'\Sigma b,$$

under the constraints

$$b'\mu = \mu_0, \qquad b'1_N = 1,$$

where 1_N is the column vector of length N whose elements are equal to 1 and $\mu_0 \in \mathbf{R}$ is a constant. The Lagrange function is

$$L(b, \lambda_1, \lambda_2) = \frac{1}{2}\, b'\Sigma b + \lambda_1(\mu_0 - b'\mu) + \lambda_2(1 - b'1_N),$$

where $\lambda_1, \lambda_2 \in \mathbf{R}$ are the Lagrange multipliers. The solution of the equation[2]

$$0 = \frac{\partial}{\partial b}\, L(b, \lambda_1, \lambda_2) = \Sigma b - \lambda_1 \mu - \lambda_2 1_N.$$

is

$$b = \Sigma^{-1}(\lambda_1 \mu + \lambda_2 1_N).$$

To get λ_1 and λ_2 we need to solve the equations

$$\begin{cases} \mu_0 &= b'\mu &= \lambda_1 \mu'\Sigma^{-1}\mu &+ \lambda_2 \mu'\Sigma^{-1} 1_N, \\ 1 &= b'1_N &= \lambda_1 1_N'\Sigma^{-1}\mu &+ \lambda_2 1_N'\Sigma^{-1} 1_N. \end{cases}$$

Denoting $\alpha = \mu'\Sigma^{-1}\mu$, $\beta = 1_N'\Sigma^{-1}1_N$, and $\gamma = 1_N'\Sigma^{-1}\mu$, we get

$$\lambda_1 = \frac{\beta\mu_0 - \gamma}{\alpha\beta - \gamma^2}, \qquad \lambda_2 = \frac{\alpha - \gamma\mu_0}{\alpha\beta - \gamma^2}.$$

Then, the vector of the portfolio weights is

$$b = \frac{1}{e}\, \Sigma^{-1} \left[\, (\alpha 1_N - \gamma\mu) + \mu_0\, (\beta\mu - \gamma 1_N) \right].$$

References

Markowitz, H. (1952), 'Portfolio selection', *J. Finance* **7**, 77–91.

Markowitz, H. (1959), *Portfolio Selection*, Wiley, New York.

[2] We have that $\frac{\partial}{\partial b}\, b'\Sigma b = (\Sigma + \Sigma')b$, and for symmetric matrices $(\Sigma + \Sigma')b = 2\Sigma b$.