Lecture Notes 9

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1 Markowitz Bullets

A Markowitz bullet is a scatter plot of points, where each point corresponds to a portfolio, the x-coordinate of a point is the standard deviation of the return of the portfolio, and the y-coordinate of a point is the expected return of the portfolio. The scatter point is called a bullet because the boundary of the scatter plot is a part of a hyperbola, and thus its shape resembles the shape of a bullet.

Figure 1 plots a collection of portfolios which are obtained from two risky assets. The expected retuns of the assets are 2 and 1.5. The standard deviations are 2 and 1. The correlation between the risky assets varies from -1 to 1. Panel (a) shows long only portfolios. The blue wedge on the left shows all portfolios that can be obtained when correlation is -1. The orange vector on the right shows all portfolios that can be obtained when correlation is 1. When correlation is -1, then there exists a portfolio with zero variance. The portfolio with zero variance should have the same return as the risk free rate, to exclude arbitrage. Panel (b) shows portfolios thatcan be obtained from the two risky assets when shorting is allowed. The weight of a portfolio can vary between -0.5 and 1.5.

Figure 2 shows portfolios obtained from three risky assets as a blue area. The three risky assets are shown as orange points. The correlations between the risky assets are 0.2, 0.5, and 0.6. Panel (a) shows all long only portfolios and panel (b) shows portfolios when shorting is allowed with restrictions. The shapes of the blue areas are irregular but the left boundaries are parts of hyperbolas.

Figure 3 shows a Markowitz bullet of long only portfolios, and it includes risk free rate and borrowing. Panel (a) shows as a blue curve long only portfolios whose components are two risky assets with correlation -0.4. The green point shows the minimum variance portfolio. The black point shows the risk free investment whose gross return is 1.1 and the variance is zero.

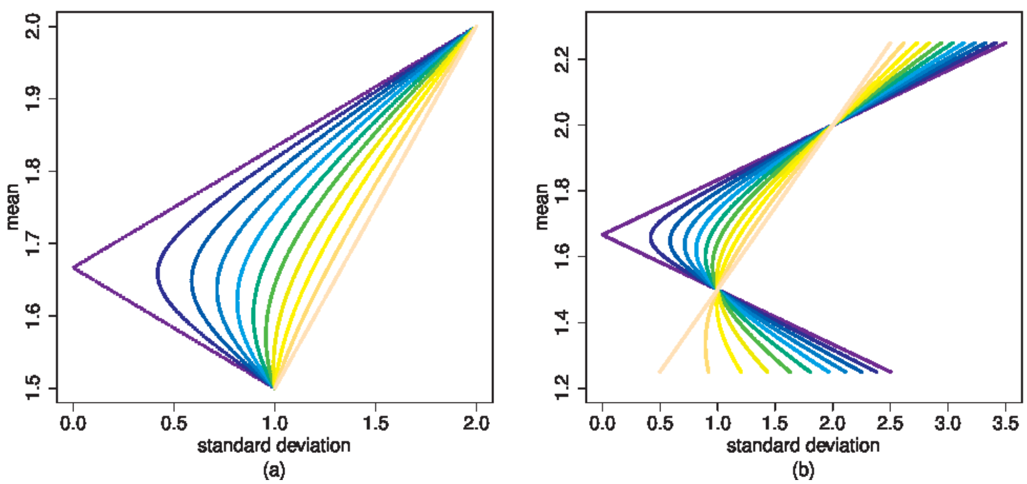

Figure 1: *Markowitz bullets: Portfolios of two risky assets when correlation varies.* (a) Shown are long only portfolios that can be obtained from two risky assets when correlation between the risky assets varies between -1 and 1. (b) Shorting is allowed with restrictions.

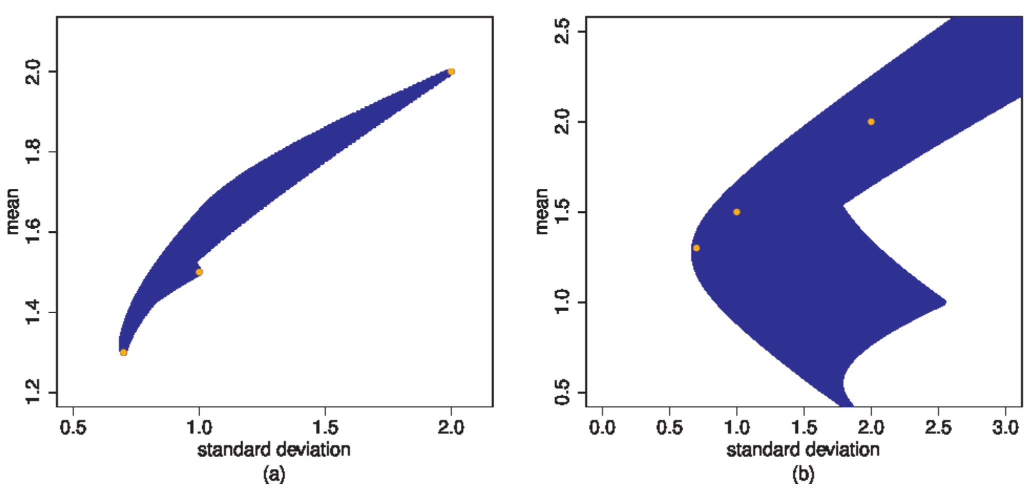

Figure 2: *Markowitz bullets: Portfolios of three risky assets.* (a) Long only portfolios that can be obtained from three risky assets. (b) Portfolios when showring is allowed.

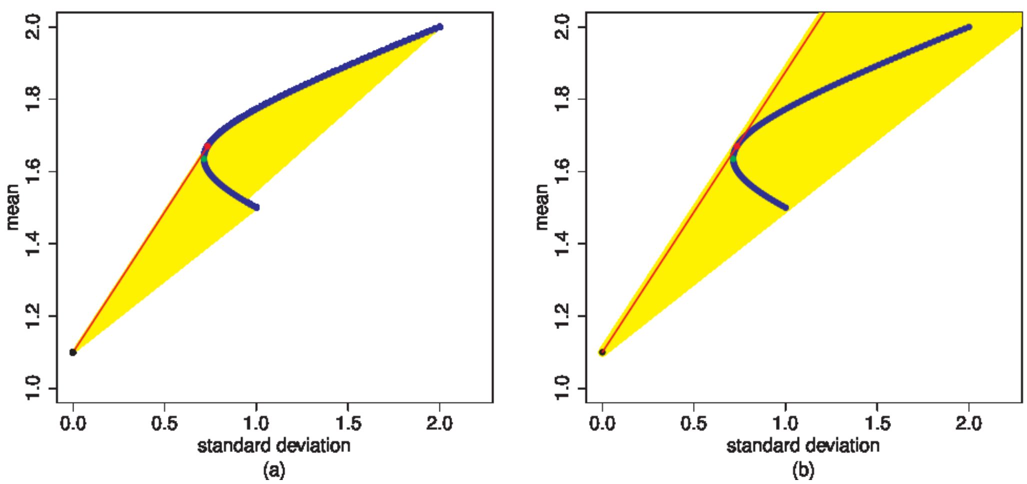

Figure 3: *Markowitz bullet: Long only portfolios and leveraging.* Panel (a) shows long only portfolios for two risky assets and the risk free investment. Panel (b) shows portfolios for two risky assets and the risk free investment when the weight of the risk free investment is allowed to be negative, which means the borrowing is allowed.

The red point shows the tangency portfolio. The red line joining the risk free investment and the tangency portfolio corresponds to the long only portfolios whose components are the risk free investment and the tangency portfolio. The yellow area corresponds to the long only portfolios whose components are the risk free investment and one of the portfolios on the blue curve; these are the all possible long-only portfolios. Panel (b) shows portfolios for two risky assets and the risk free investment when the weight of the risk free investment is allowed to be negative, which amounts to allowing leveraging by borrowing.

We ca use Figure 3 to define the concepts of the minimum variance portfolio, the tangency portfolio, and the efficient frontier.

1. The minimum variance portfolio is the portfolio of risky assest whose variance is the smallest among all portfolios of risky assets. When the risk free rate is included, then the risk free investment has the minimum variance zero.

2. Efficient frontier is the collection of those portfolio vectors that have the expected return greater or equal than the expected return of the minimum variance portfolio:

In Figure 3(a) the efficient frontier without the risk free rate is the part of the blue curve going upwards from the red point, but when the the risk free rate is included, then the red vector from the risk free asset to the tangency portfolio, followed by the blue curve shows the efficient frontier. The efficient frontier consists of possible portfolios a rational investor should consider, because these portfolios have a higher expected return with the same variance than other portfolios. Adding the risk free rate gives the possibility to get portfolios with a smaller standard deviation than any of the pure stock portfolios: some of the portfolios on the red vector are such that the standard error is smaller than the standard deviation of any of the pure stock portfolios.

In Figure 3(b) borrowing is allowed. The borrowed money is invested in the stocks. Now the efficient frontier is the red half line starting from the risk free investment and passing the tangency portfolio. We see that a rational investor chooses only portfolios that are a combination of the risk free investment and the tangency portfolio. The other portfolios have a smaller expected return for the same variance.

3. The tangency portfolio is a portfolio which hast the largest Sharpe ratio. Indeed, the tangency portfolio maximizes the slope of the vector drawn from the risk free asset to a pure stock portfolio. The slope of the vector from the point $(0, R_f)$ to the point $(\mathrm{sd}(R_t^p), ER_t^p)$ is equal to the Sharpe ratio $(ER_t^p - R_f)/\mathrm{sd}(R_t^p)$, where R_f is the return of the risk free asset and R_t^p is the return of a portfolio.

4. It can be argued, that the tangency portfolio, shown as the red point in the blue curve, is in fact the market portfolio, because the rational investor buys only a combination of the tangency portfolio and the risk free asset, and thus the price of the tangency portfolio is in the equilibrium equal to the price of the market portfolio.

Figure 4 plots standard deviations and means for a collection of portfolios when shorting of a stock is allowed. Panel (a) shows portfolios for two risky assets. The blue part shows the long only portfolios, the orange part shows the portfolios where the less risky stock is shorted, and the purple paart shows the portfolios where the more risky stock is shorted. The green bullet shows the minimum variance portfolio, the black bullet show the risk free investment, and the red bullet shows the tangency portfolio. Panel (b) shows portfolios of two risky assets and a risk free investment when the weight of the risk free investment is allowed to be negative, which amounts to allowing leveraging by borrowing.

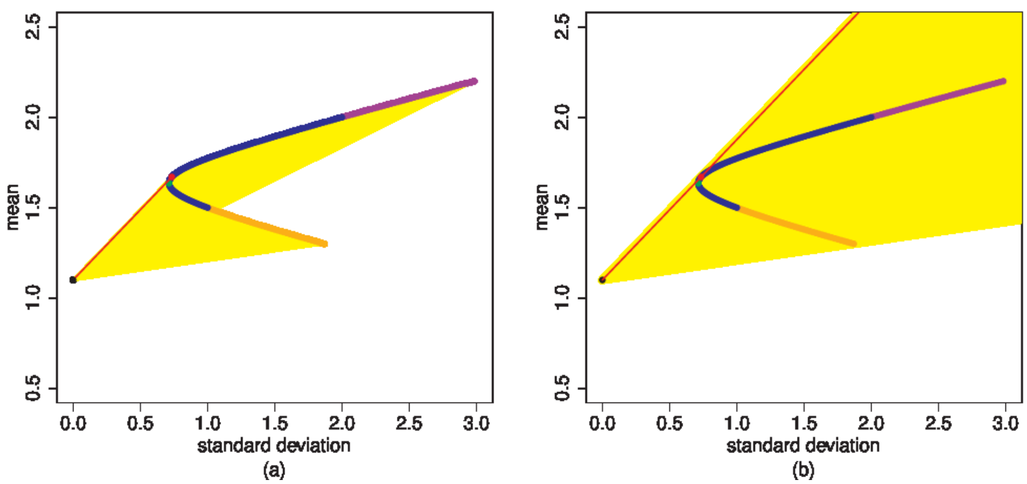

Figure 4: *Markowitz bullet: Shorting and leveraging.* Panel (a) shows portfolios for two risky assets, and for two risky assets and the risk free investment. Panel (b) shows portfolios of two risky assets and a risk free investment when the weight of the risk free investment is allowed to be negative, so that borrowing is allowed.

Figure 5 shows how increasing the number of basis assets makes the Markowitz bullet larger. The blue hyperbola shows portfolios that can be obtained from two risky assets, the green area shows portfolios that can be obtained from three risky assets, and the yellow area shows portfolios that can be obtained from four risky assets. The orange points show the risky assets. The covariance of the risky assets are zero.

2 Diversification

The expected return of a portfolio is determined by the expected returns of the basis assets, but the risk of the return distribution depends on the joint distribution of the returns of the basis assets.

Let us illustrate how the risk of a portfolio depends on the joint distribution of the basis assets by using the case of two basis assets as an example. We use now the variance of the portfolio return as the measure of the risk of the portfolio. Let R^1_{t+1} and R^2_{t+1} be the gross returns of two basic assets. Let us assume the $\mathrm{Var}(R^1_{t+1}) = \mathrm{Var}(R^2_{t+1}) = \sigma^2$ and $\mathrm{Cor}(R^1_{t+1}, R^2_{t+1}) = \rho$. Then the variane of the portfolio returns is

$$\mathrm{Var}\left(bR^1_{t+1} + (1-b)R^2_{t+1}\right) = b^2\sigma^2 + (1-b)^2\sigma^2 + 2b(1-b)\sigma^2\rho.$$

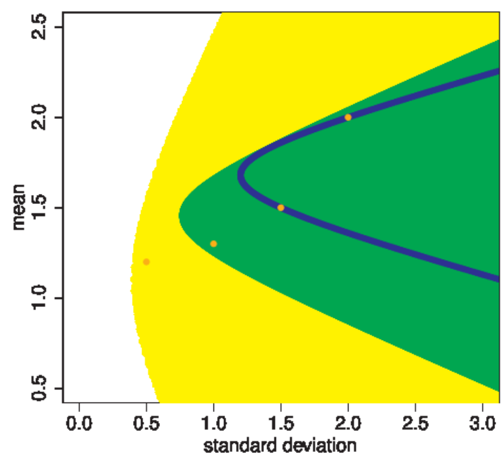

Figure 5: *Markowitz bullet: Uncorrelated assets.* Markowitz bullets are shown for an increasing number of assets: blue curve shows portfolios from two risky assets, the green area portfolios of three risky assets, and the yellow area portfolios of four risky assets, when the risky assets are uncorrelated.

Figure 6 shows the function $(\rho, b) \mapsto b^2 + (1-b)^2 + 2b(1-b)\rho$, where we have chosen the variance of the portfolio components to be $\sigma^2 = 1$. The variance of the portfolio becomes smaller when $\rho \to -1$. When $0 \le b \le 1$, then variance of the portfolio is smaller than one, otherwise it is larger than one. Thus, the variance of the portfolio is smaller than the variance of the components when $0 \le b \le 1$, and the reduction in the variance is greatest when portfolio components are anticorrelated.

In the case of two basic assets the variance of the portfolio return can be close to zero when the two assets are anticorrelated. The variance can also be close to zero, when we have a large number of uncorrelated basic assets. Consider N assets $S^1, \ldots, S^N$. We denote the returns of the basis assets by $R_t^i = S_t^i / S_{t-1}^i$, $i = 1, \ldots, N$. Let us assume that $ER_t^i = \mu$, $\mathrm{Var}(R_t^i) = \sigma^2$, and that the returns are uncorrelated. Let the portfolio vector be $b = (1/N, \ldots, 1/N) \in \mathbf{R}^N$. Then,

$$E(b'R_t) = \mu, \qquad \mathrm{Var}(b'R_t) = \frac{\sigma^2}{N}.$$

Thus, when the number N of assets in the portfolio is large, the variance of the portfolio return is close to zero.

In practice it is difficult to find uncorrelated basis assets and it is even more difficult to find anticorrelated basis assets. However, even when the

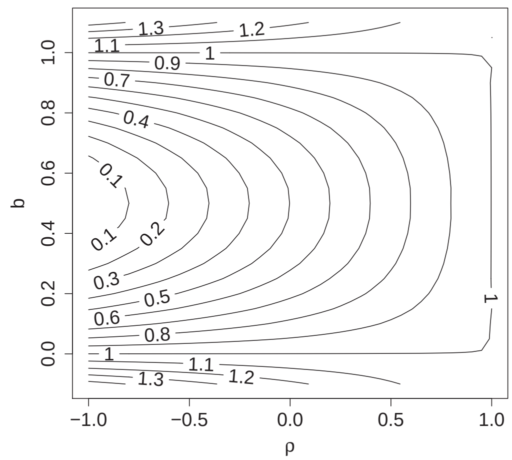

Figure 6: *Variance of the portfolio returns.* Function $(\rho, b) \mapsto b^2 + (1-b)^2 + 2b(1-b)\rho$ is shown. This function shows the variance of the porttfolio when the portfolio components have variance one, correlation ρ, and the weight of the components is b and $1-b$.

basis assets are correlated it is possible to decrease the risk of the portfolio by allocating the portfolio weights skillfully among the basis assets.